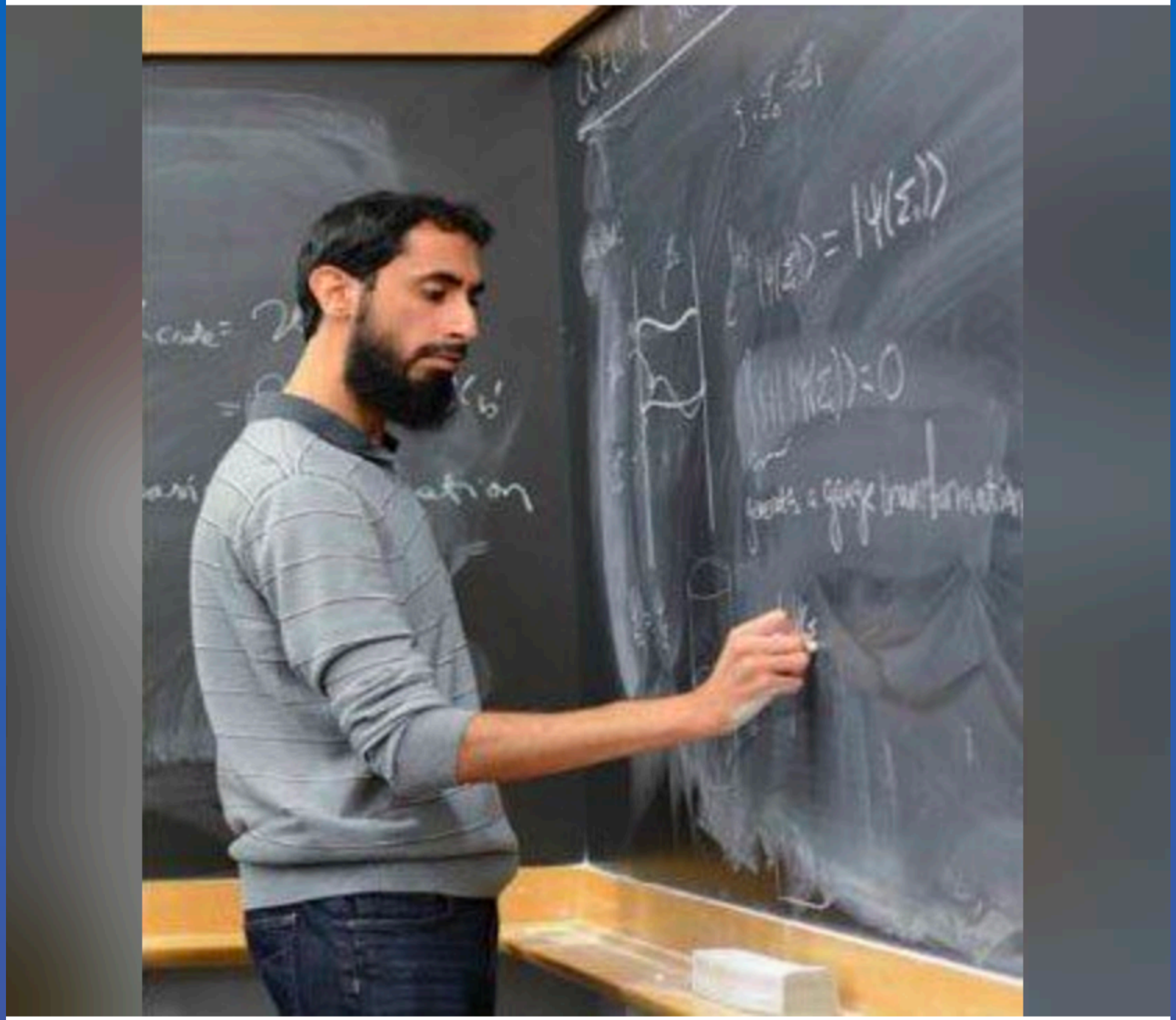
Holography and Quantum Error Correction

Part III:

Quantum Error Correction

Ahmad Almheiri

Are you really from Abu Dhabi?

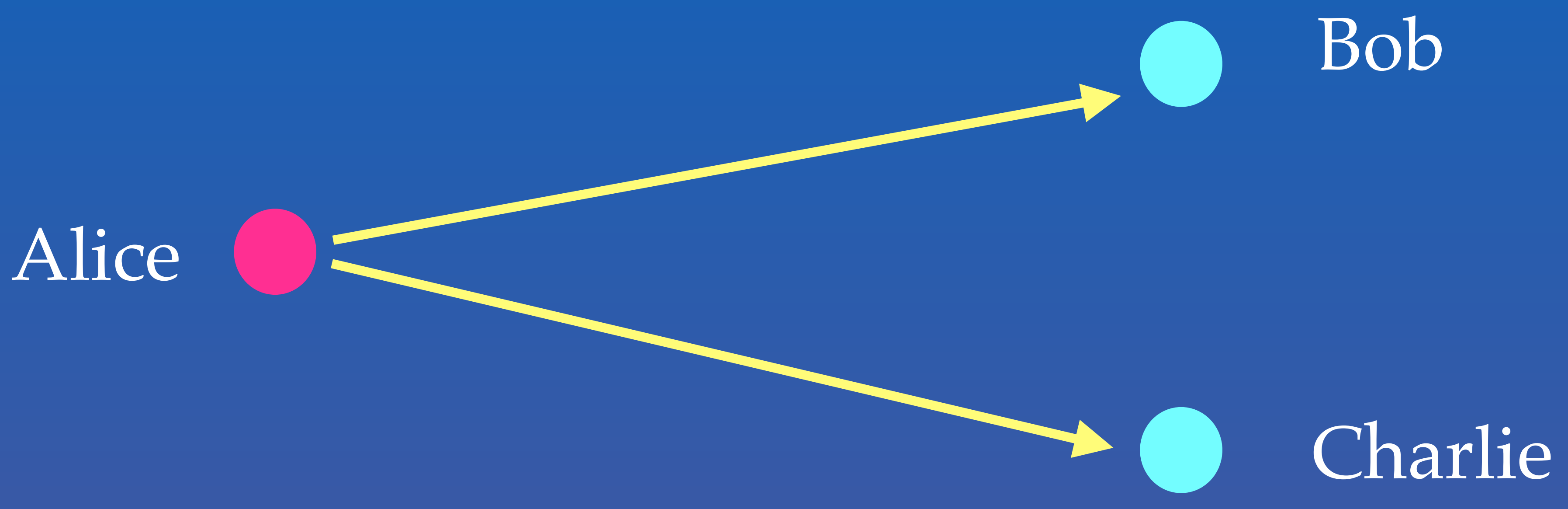


آیا شما واقعا اهل ابوظبی هستید؟

امسال جایزه سه میلیون دلاری فیزیک بنیادی Breakthrough Prize به احمد المهیری
Ahmad Almheiri
فیزیکدان اماراتی ساکن امریکا تعلق گرفت. او که پژوهشگر موسسه مطالعات پیشرفته پرینستون است این جایزه را برای کارهای پیشتازانه اش در محاسبه محتوای اطلاعات و تابش سیاه چاله ها در یافت کرده است. المهیری که در یک خانواده پرجمعیت با ۹ خواهر و یک برادر بزرگ شده تا کلاس نهم دانشجوی درسخوانی نبوده و تنها پس از گزارش این کاهلی به مادرش تصمیم به درس خواندن می گیرد و بعد از گرفتن لیسانس فیزیک به امریکا می رود. برای رسیدن به این درجات لازم نیست شخص از بچگی نابغه باشد، و مهم تر از آن لازم نیست حتما غربی، هندی، ژاپنی و چینی یا ایرانی باشد. عرب ساکن امارات هم می تواند به بزرگی برسد. خودش گفته است خیلی ها وقتی با او سر صحبت را باز می کنند اولین سوالی که ازش می پرسند این است که آیا واقعا شما اهل ابوظبی هستید؟

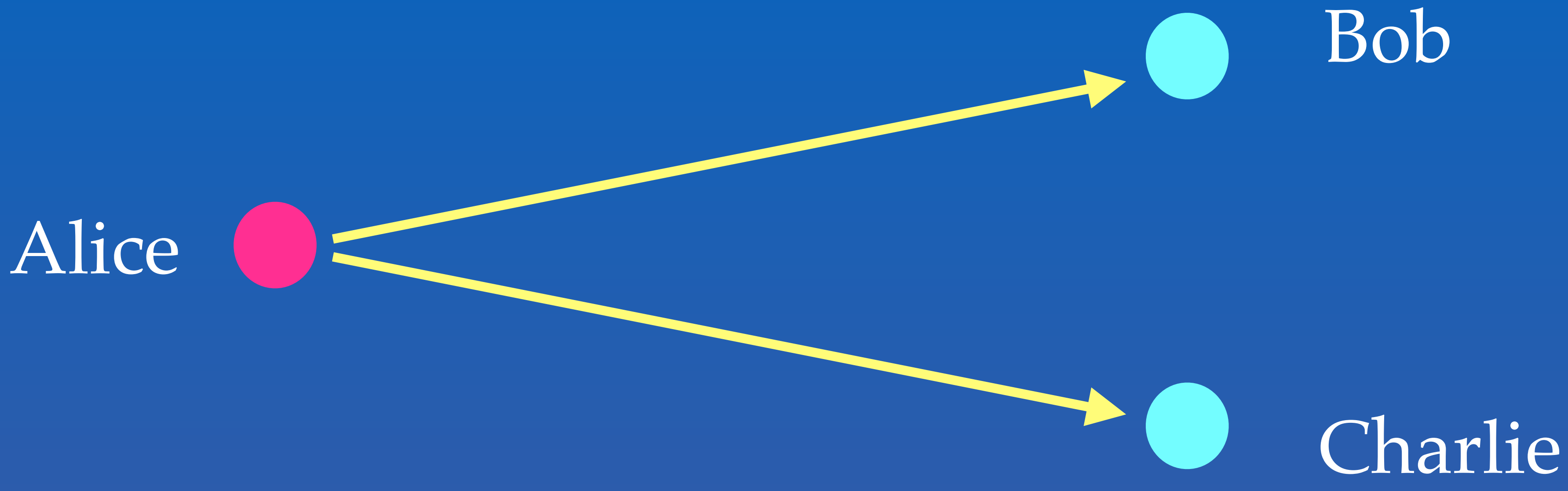
Quantum State Sharing

A (1,2) scheme



$$|0\rangle \longrightarrow |\bar{0}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|1\rangle \longrightarrow |\bar{1}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

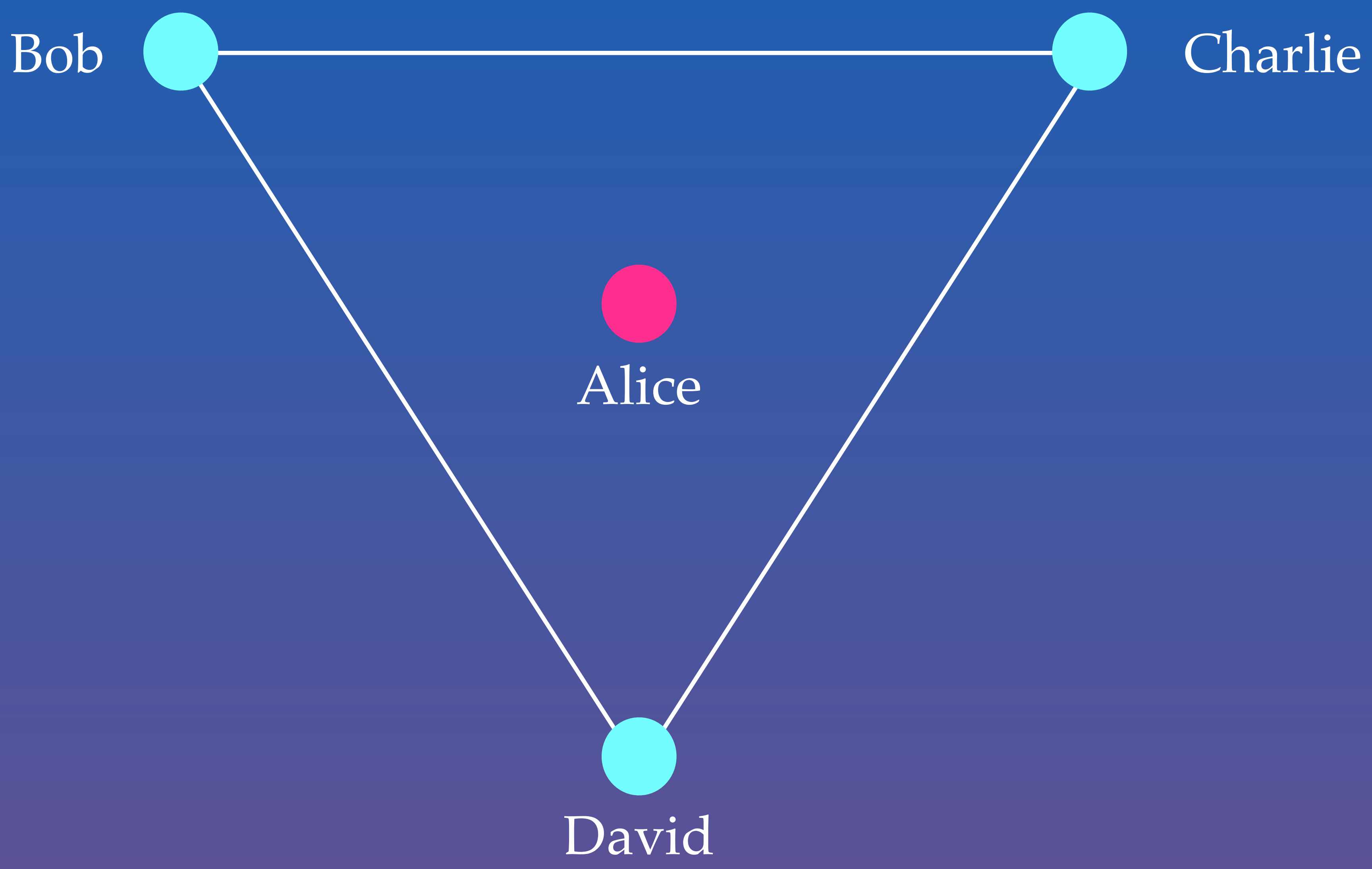


$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \longrightarrow |\overline{\psi}\rangle = \alpha\,|\overline{0}\rangle + \beta\,|\overline{1}\rangle$$

Bob and Charlie can collaborate
to retrieve the original state $|\psi\rangle$

A more complicated schemes: (2,3)

Quantum Secret Sharing = Quantum Erasure Code



For a moment forget the normalization factor $\frac{1}{\sqrt{3}}$

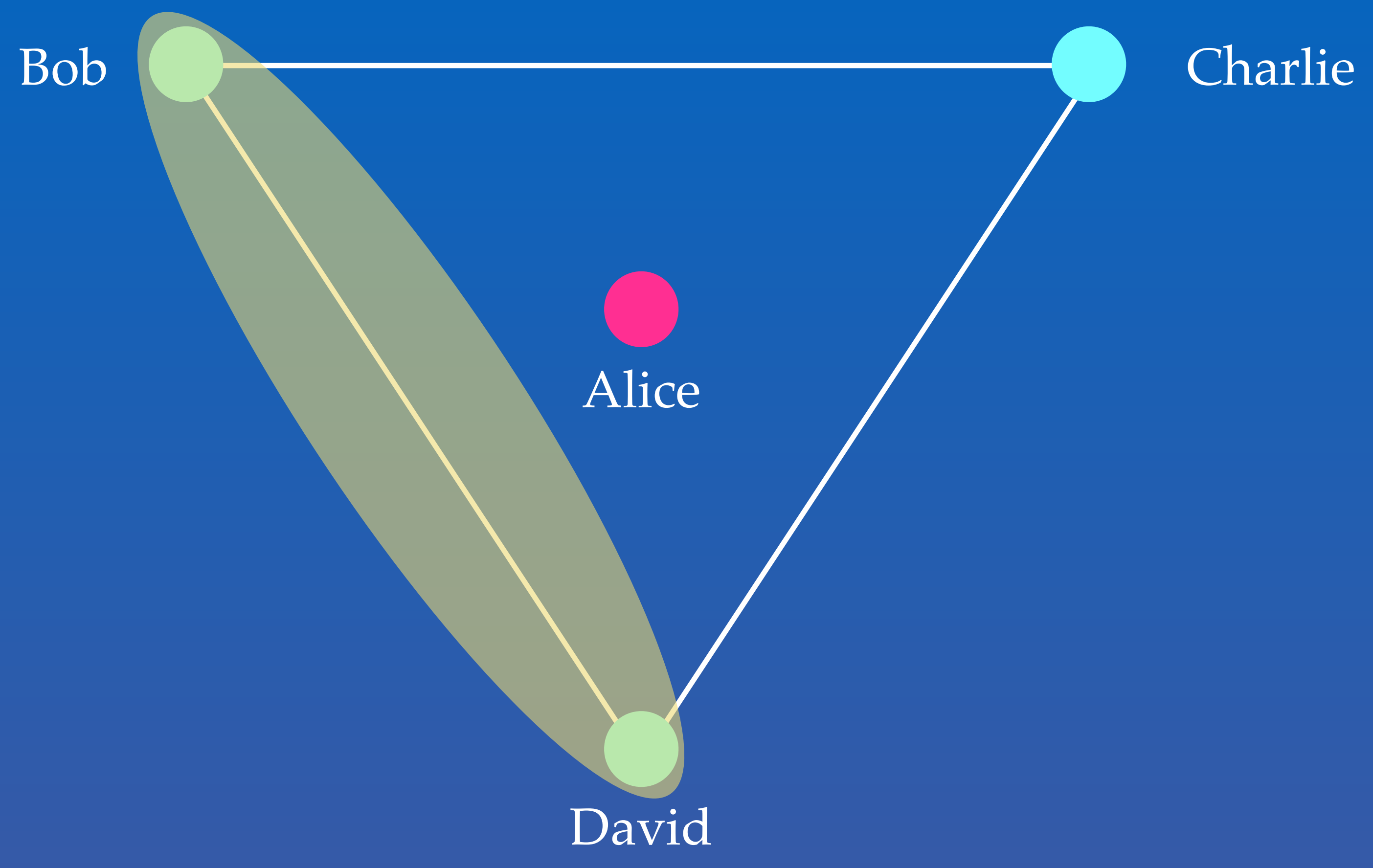
$$|0\rangle \longrightarrow \quad |\bar{0}\rangle = |000\rangle + |111\rangle + |222\rangle$$

$$|1\rangle \longrightarrow \quad |\bar{1}\rangle = |012\rangle + |120\rangle + |201\rangle$$

$$|2\rangle \longrightarrow \quad |\bar{2}\rangle = |021\rangle + |102\rangle + |210\rangle$$

For qubits this is not possible

How a state is retrieved?



$$|\bar{0}\rangle = |000\rangle + |111\rangle + |222\rangle$$

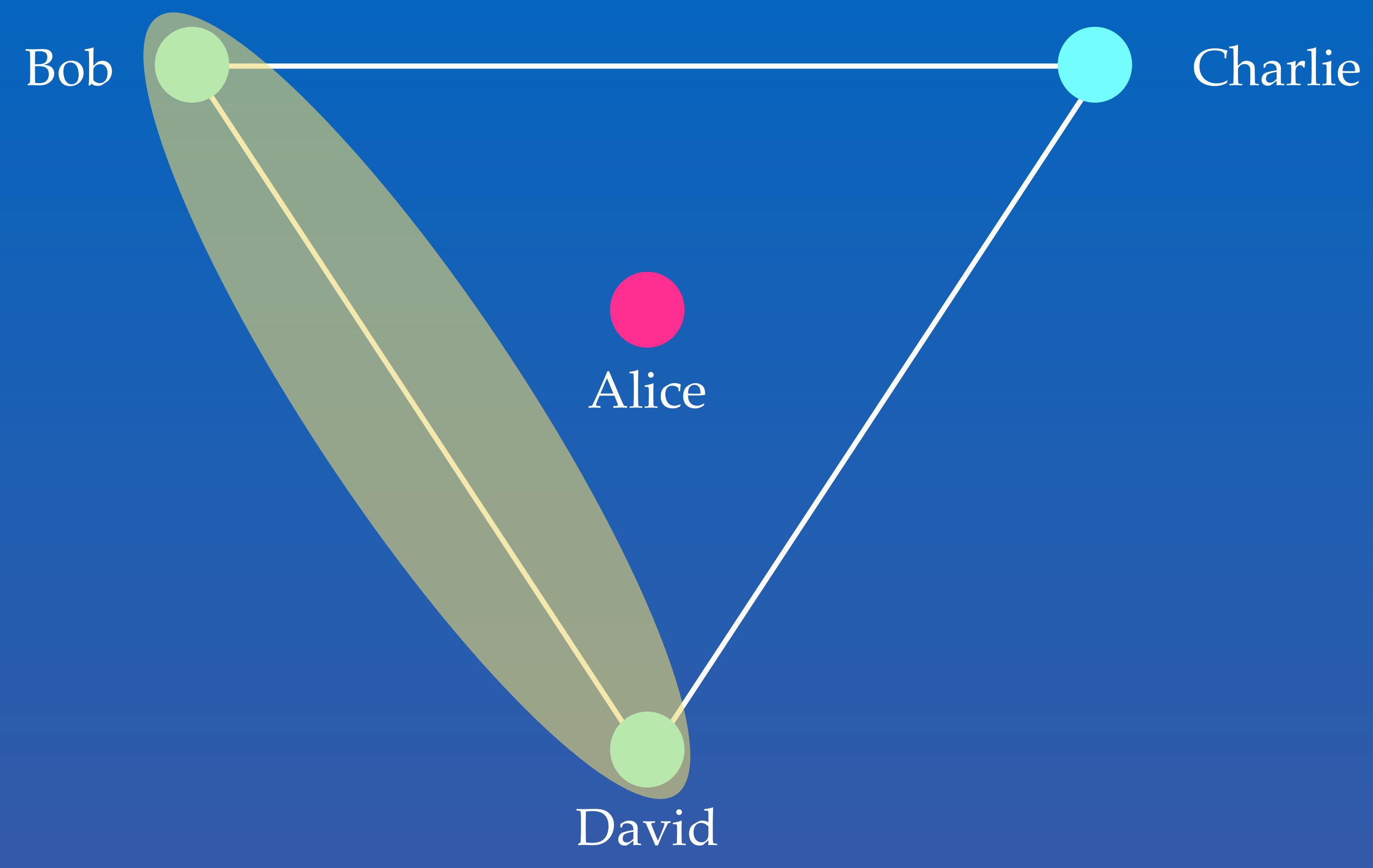
$$C_{12}$$

$$\longrightarrow |000\rangle + |121\rangle + |212\rangle$$

$$C_{21}$$

$$\longrightarrow |000\rangle + |021\rangle + |012\rangle$$

$$= |0\rangle \otimes \Big(|00\rangle + |21\rangle + |12\rangle\Big)$$



$$|\bar{1}\rangle = |012\rangle + |120\rangle + |201\rangle$$

$$\longrightarrow |012\rangle + |100\rangle + |221\rangle \qquad C_{12}$$

$$\longrightarrow |112\rangle + |100\rangle + |121\rangle \qquad C_{21}$$

$$= |1\rangle \otimes \Big(|12\rangle + |00\rangle + |21\rangle \Big)$$

$$|0\rangle \longrightarrow \quad |\bar{0}\rangle = |000\rangle + |111\rangle + |222\rangle$$

$$|1\rangle \longrightarrow \quad |\bar{1}\rangle = |012\rangle + |120\rangle + |201\rangle$$

$$|2\rangle \longrightarrow \quad |\bar{2}\rangle = |021\rangle + |102\rangle + |210\rangle$$

$$Tr_{23}(|\bar{i}\rangle\langle\bar{j}|) = \delta_{ij}\ I$$

$$\downarrow$$

$$Tr_{23}(|\overline{\psi}\rangle\langle\overline{\psi}|) = \frac{1}{3}I$$

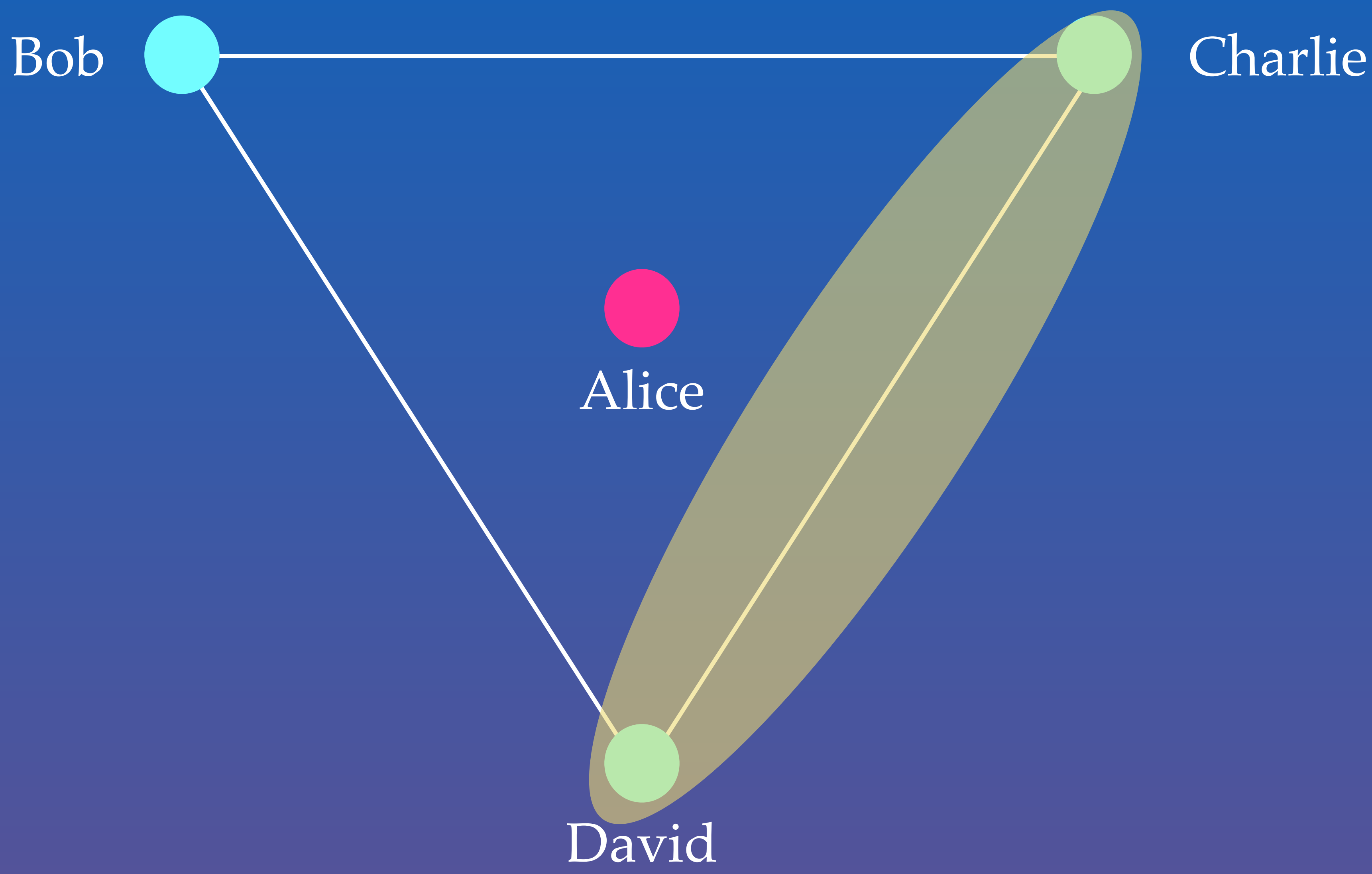
Neither of the player has any idea of the state shared by Alice!

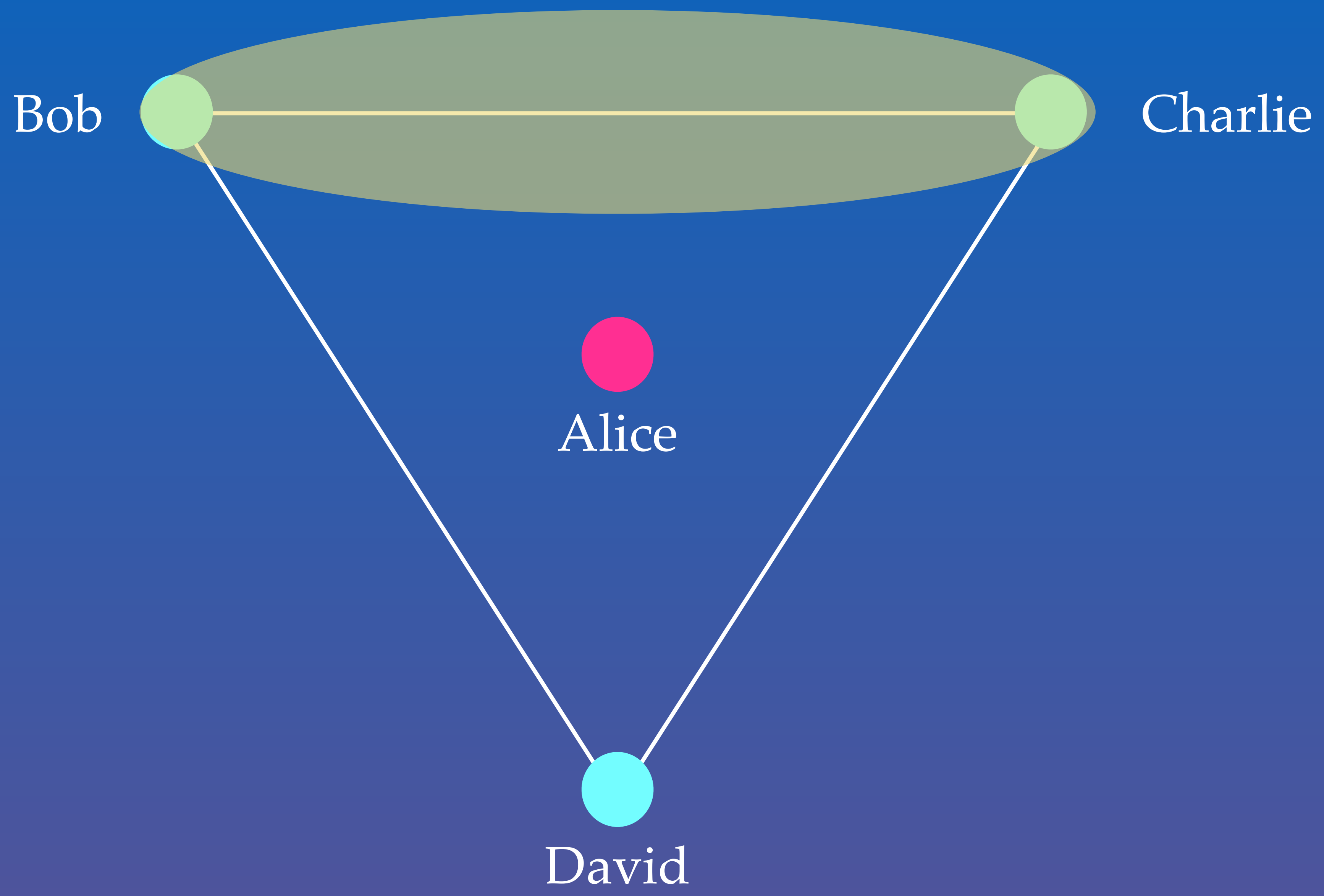
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \longrightarrow |\overline{\psi}\rangle = \alpha|\overline{0}\rangle + \beta|\overline{1}\rangle + \gamma|\overline{2}\rangle$$

$$Tr_{23}(|\bar{i}\rangle\langle\bar{j}|) = \delta_{ij}\ I$$

$$Tr_{23}(|\overline{\psi}\rangle\langle\overline{\psi}|) = \alpha_i\alpha_j^*\delta_{ij}\ I = I$$

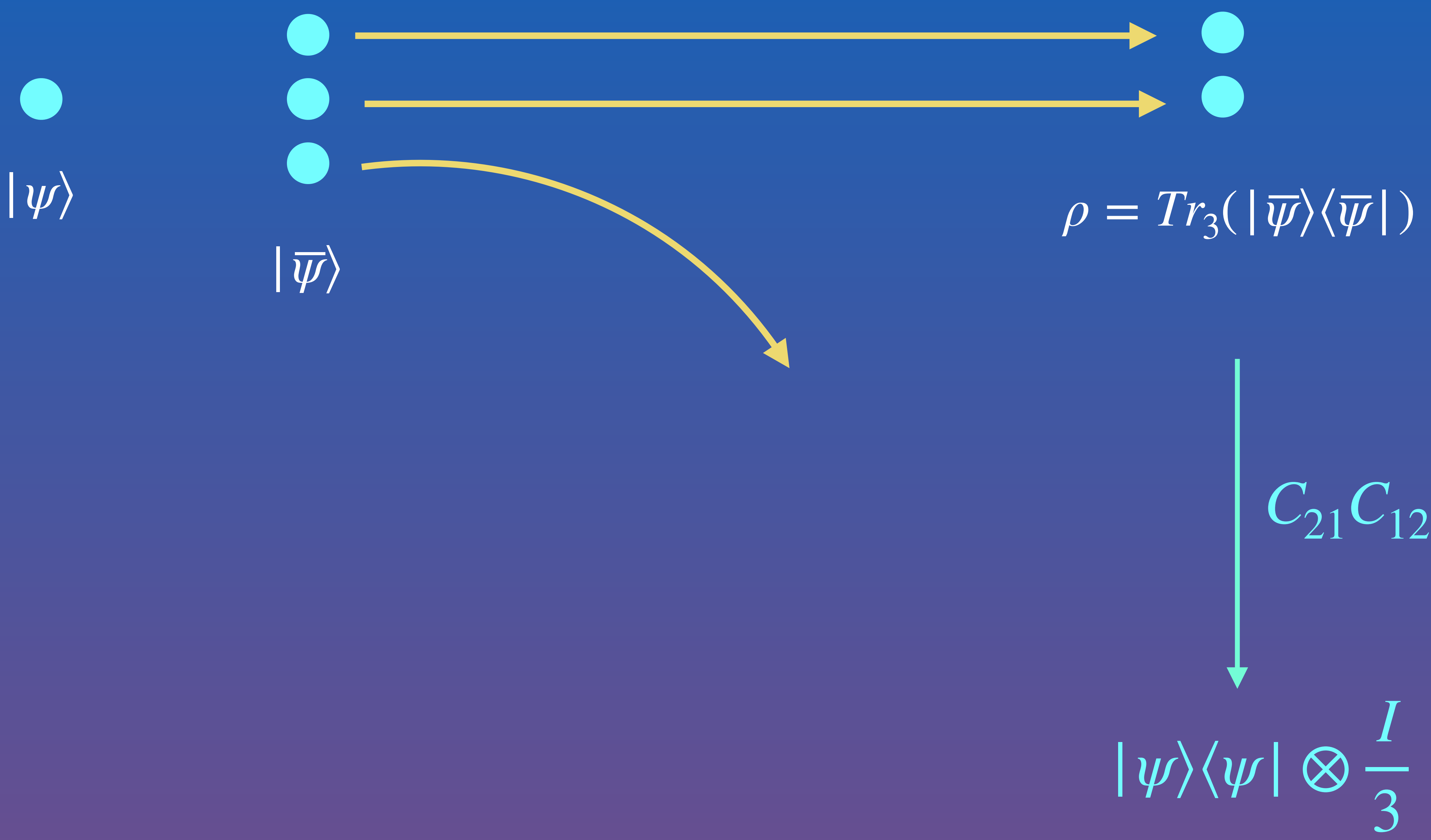
We have ignored normalization.


Bob
Charlie
Alice
David



This is the simplest example of holography.
The state in the bulk is distributed to the boundary.

If one of the qubits is lost, the state can still be recovered!



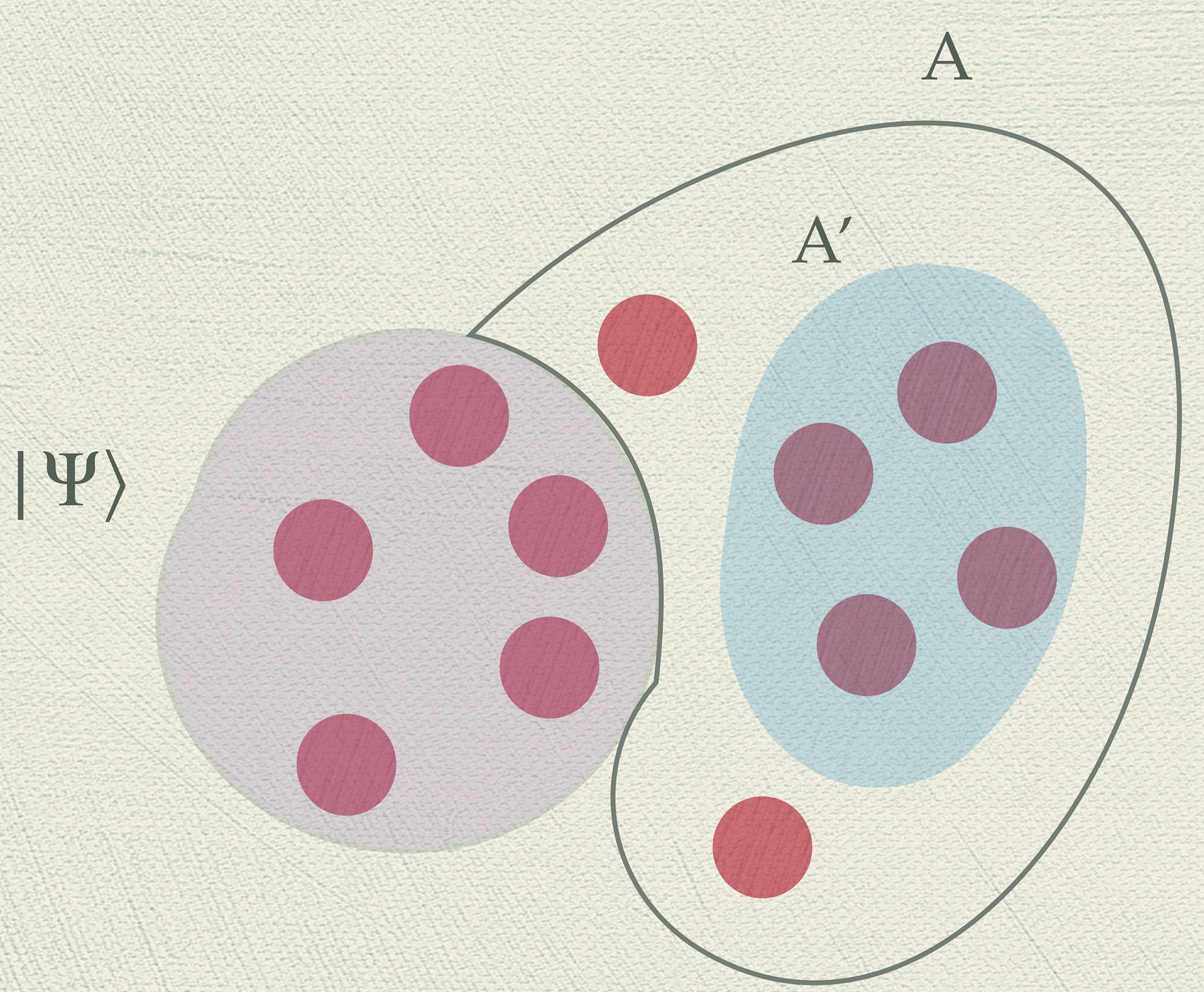
Now we consider a new but related subject.

Absolutely Maximally Entangled States

These are the states which have the highest amount of entanglement.

Definition of AME

$|\Psi\rangle$ is an AME if for every half partition, it is maximally mixed.

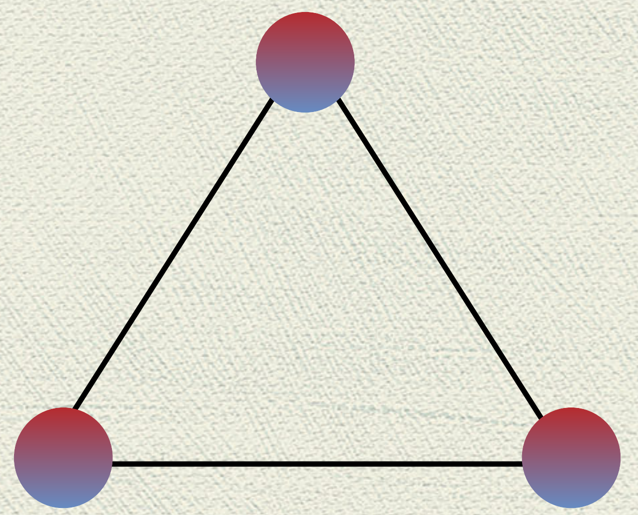


The simplest example

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\rho_1 = \frac{I}{2} \qquad \rho_2 = \frac{I}{2}$$

A 3-qubit example

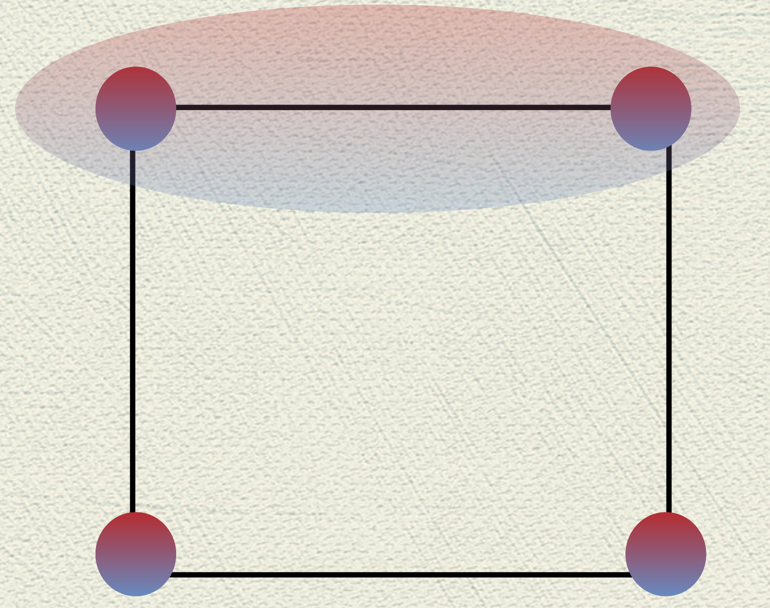


$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$\rho_1 = \frac{I}{2} \qquad \rho_2 = \frac{I}{2} \qquad \rho_3 = \frac{I}{2}$$

This is NOT a good example

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$



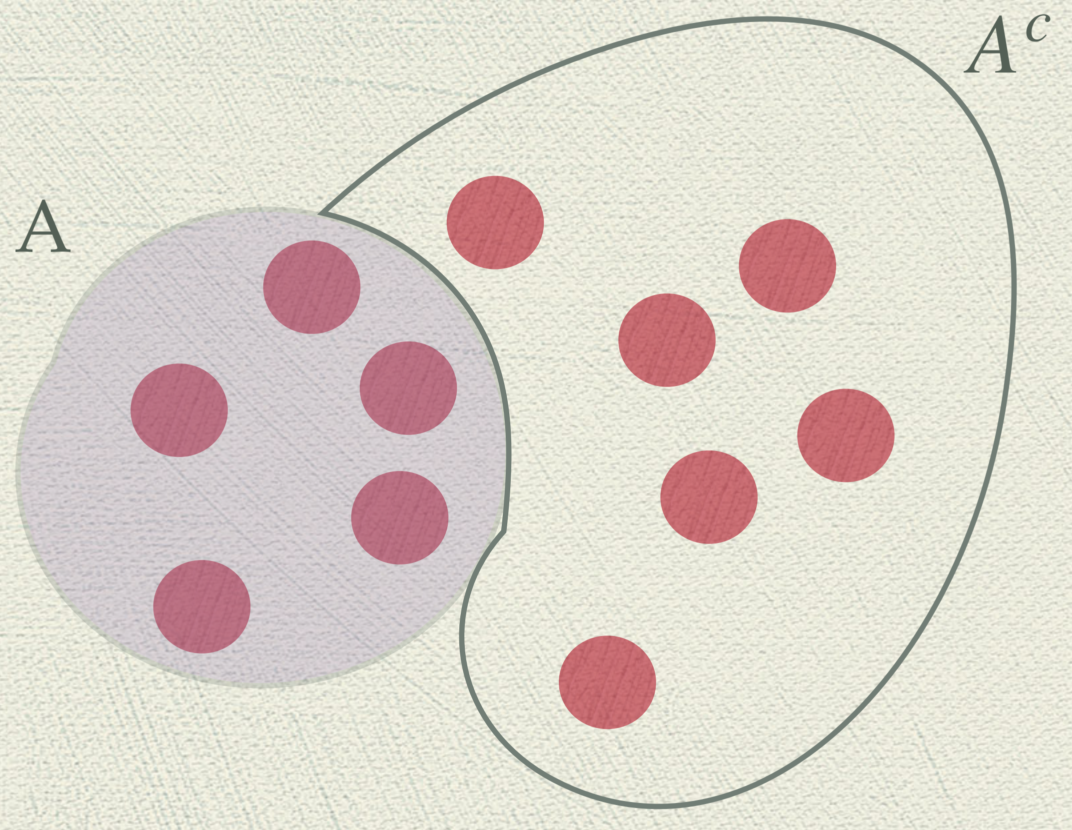
$$\rho_{12} = \frac{I}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \neq \frac{1}{4}I$$

In fact there are no 4-qubit AME states.

The general form of AME:

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |I\rangle_A |\phi_I\rangle_{A^c}$$

a basis for A.



$$\langle \phi_I | \phi_J \rangle \propto \delta_{I,J}$$

Dimension of A = d

Dimension of $A^c = d'$

An AME state of qutrits

$$|\Psi\rangle = \sum_{i,j=0}^{2} |i, j, i+j, i+2j\rangle$$

$$|\psi\rangle = |0000\rangle + |0112\rangle + |0221\rangle$$

$$+|1011\rangle + |1120\rangle + |1202\rangle$$

$$+|2022\rangle + |2101\rangle + |2210\rangle$$

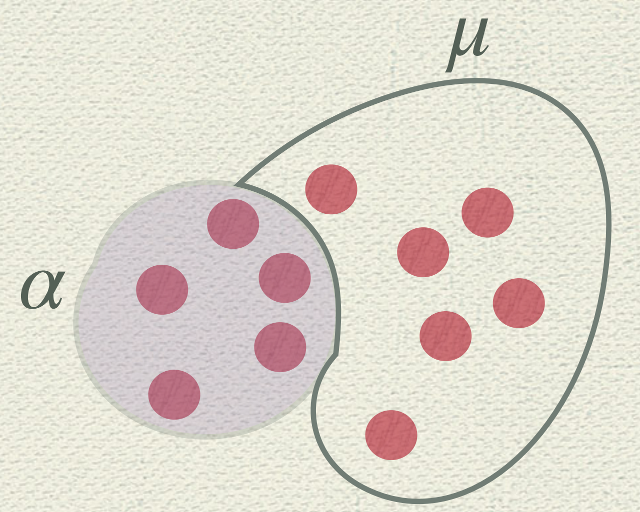
$$|\psi\rangle = \frac{1}{2} \sum_{i,j,k,l} T_{ijkl} |i, j, k, l\rangle$$

Let's write an AME state as follows:

$$|T\rangle = \sum_{i,j,k,l} T_{ijkl} |i,j,k,l\rangle$$

Since $|T\rangle$ is AME , for every half partition of indices we have :

$$T_{\alpha\mu} T^*_{\beta,\mu} \propto \delta_{\alpha,\beta}$$



Note : $T_{ijk\ldots l} \equiv T_{\alpha,\mu}$

Such a tensor is called a perfect tensor.

Therefore:

AME states = Perfect Tensors

Do we always have AME states? Or Perfect Tensors?

The answer depends on d and n.

For 4 qubits, there is no perfect tensor.

For 5 qubits, there are perfect tensors.

For 4 qutrits, there are perfect tensors.

Now we consider a different but related subject.

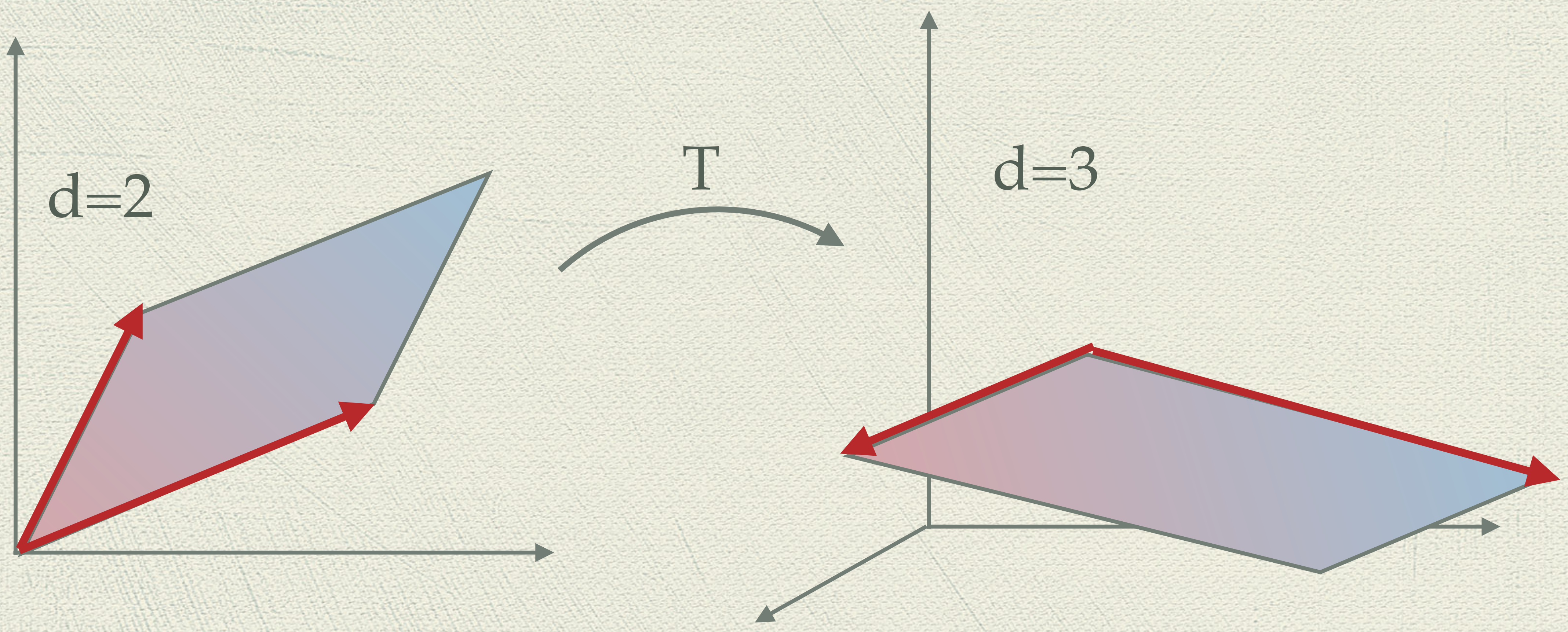
Isometries and Multi-Isometries

Definition of Isometry

A transformation which preserves the inner product.

$$T : H \longrightarrow H'$$

$$\langle x, y \rangle = \langle Tx, Ty \rangle \longrightarrow T^\dagger T = I_d$$



We must have $d \leq d'$

Difference with unitary transformation

$$T : H \longrightarrow H'$$

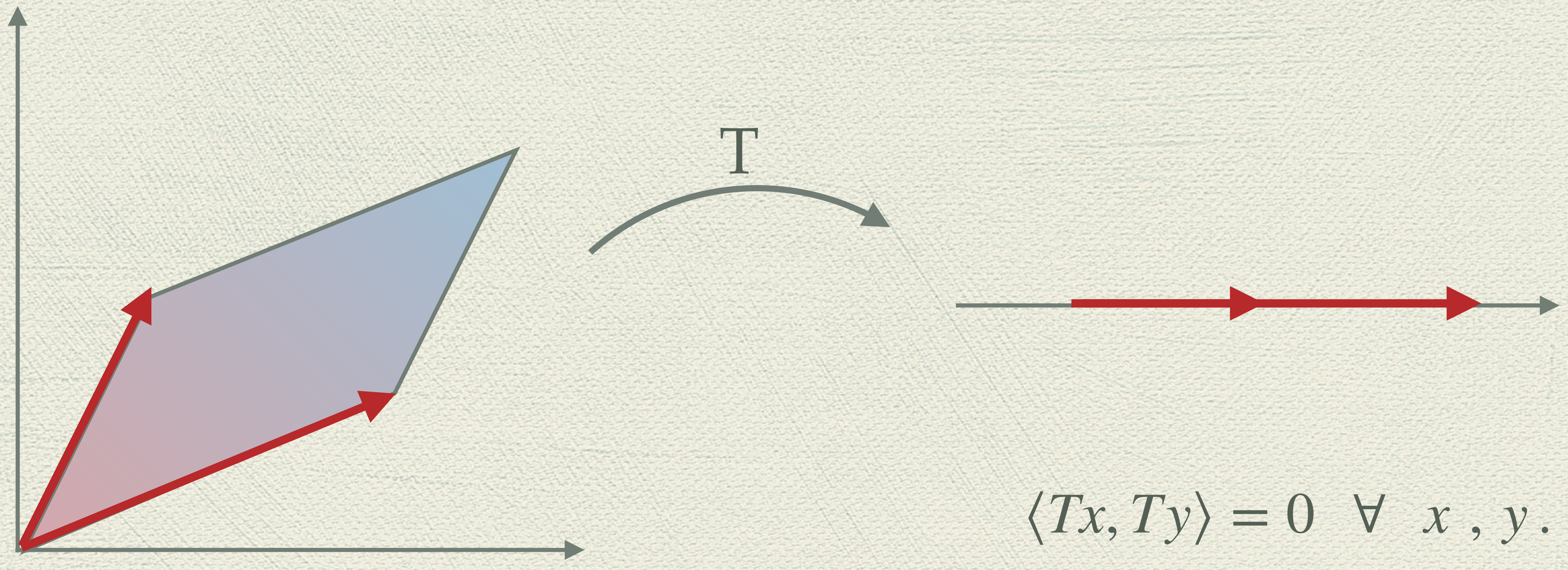
$$\langle x, y \rangle = \langle Tx, Ty \rangle \longrightarrow T^{\dagger}T = I_d$$

But

$$TT^{\dagger} \neq I_{d'}$$

A simple counter example

if $d' < d$

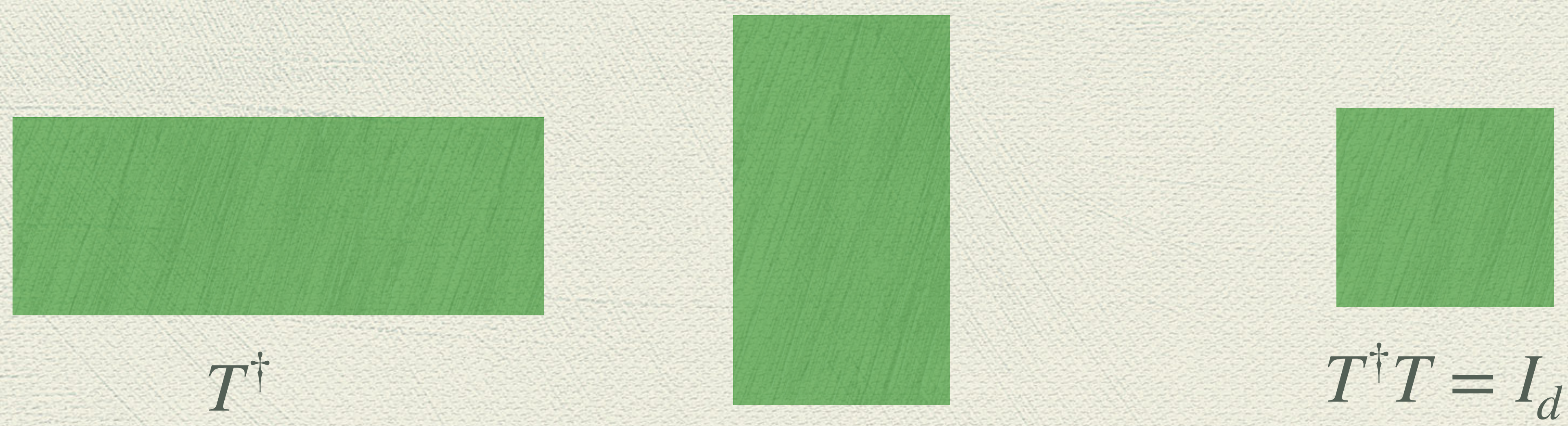


$$\langle Tx, Ty \rangle = 0 \quad \forall \; x, y.$$

The inner product cannot be preserved,
since all inner product become zero.

A Graphical proof:

$T : H \longrightarrow H'$ $\qquad d < d'$



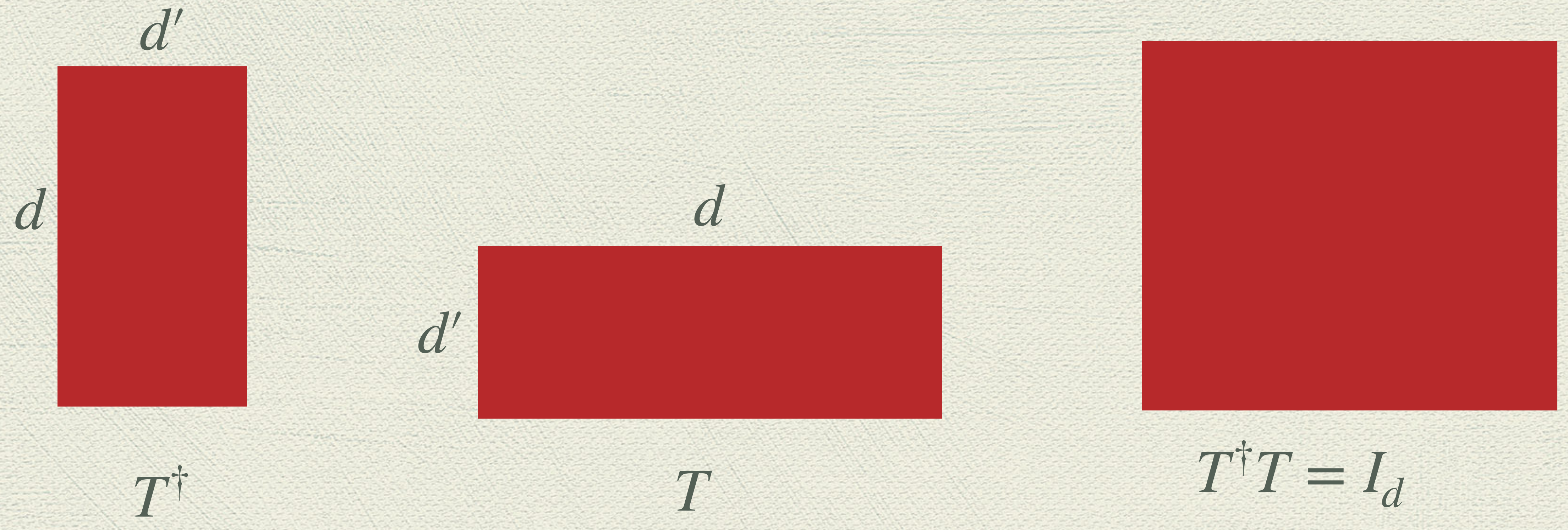
$T^\dagger$ $\qquad T$ $\qquad T^\dagger T = I_d$

d^2 equations for dd' parameters

We have solutions.

If d>d′

$$T : H \longrightarrow H'$$



d^2 equations for dd' parameters

We may not have solutions.

If $T = \sum_{\mu,\alpha} T_{\mu\alpha} |\mu\rangle\langle\alpha|$ is an isometry

then : $\sum_{\mu} T_{\mu\alpha} T^*_{\mu,\beta} \propto \delta_{\alpha\beta}$

A tensor which has this property for each partition of indices is called Muti-isometry.

Therefore:

AME states = Perfect Tensors = Multi-isometeies

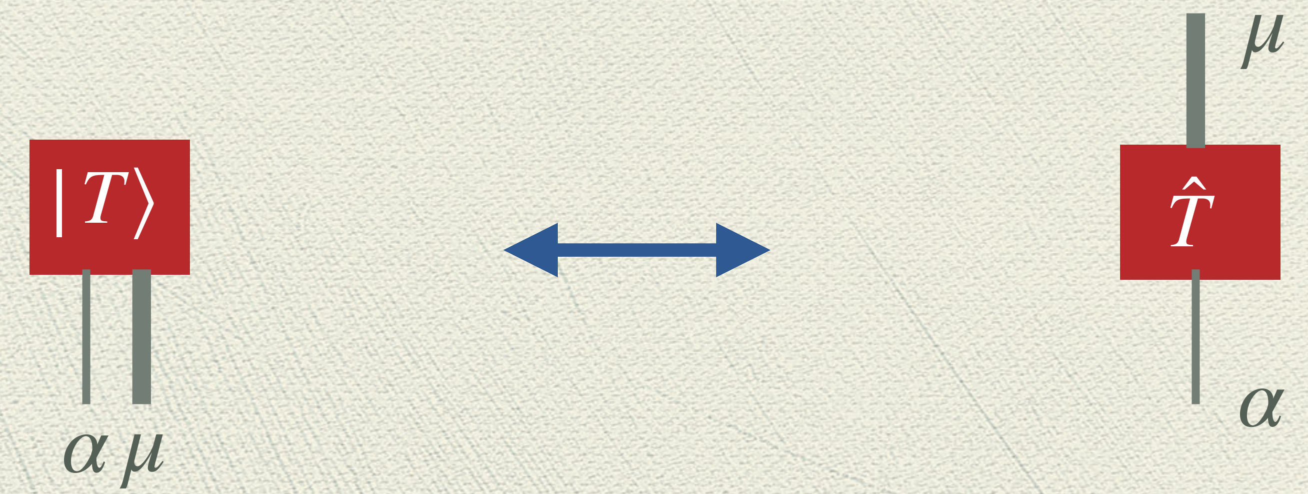
Graphical Representation

$$|T\rangle = \sum_{\mu,\alpha} T_{\mu,\alpha} |\mu, \alpha\rangle$$

$$\hat{T} = \sum_{\mu,\alpha} T_{\mu,\alpha} |\mu\rangle\langle\alpha|$$

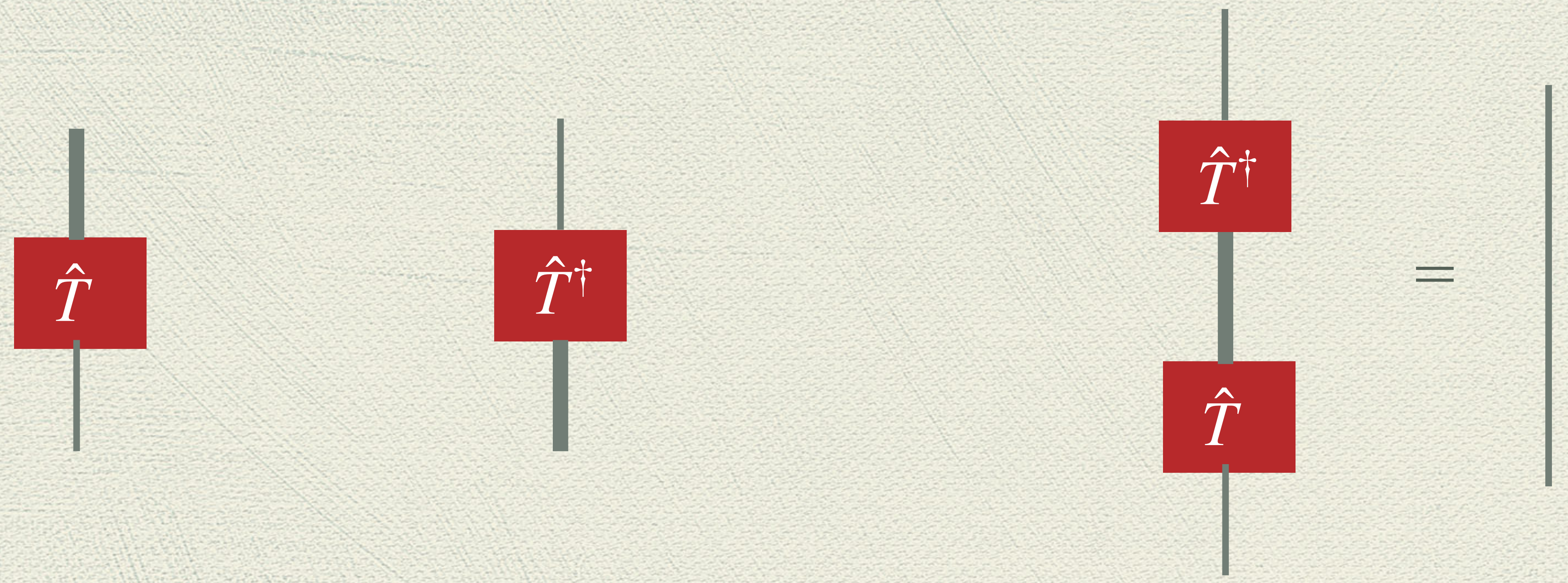
$|T\rangle$ maximally entangled

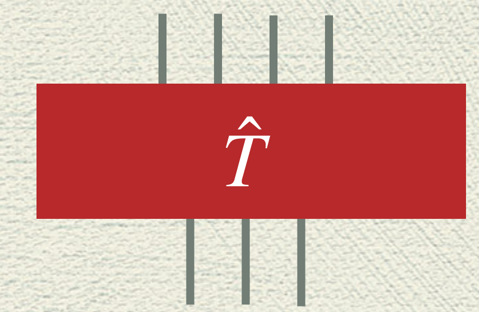
$\hat{T} \propto$ Isometry



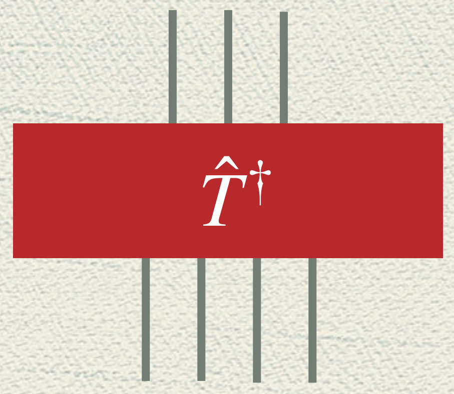
Graphical Representation

$\hat{T} \propto \text{Isometry}$

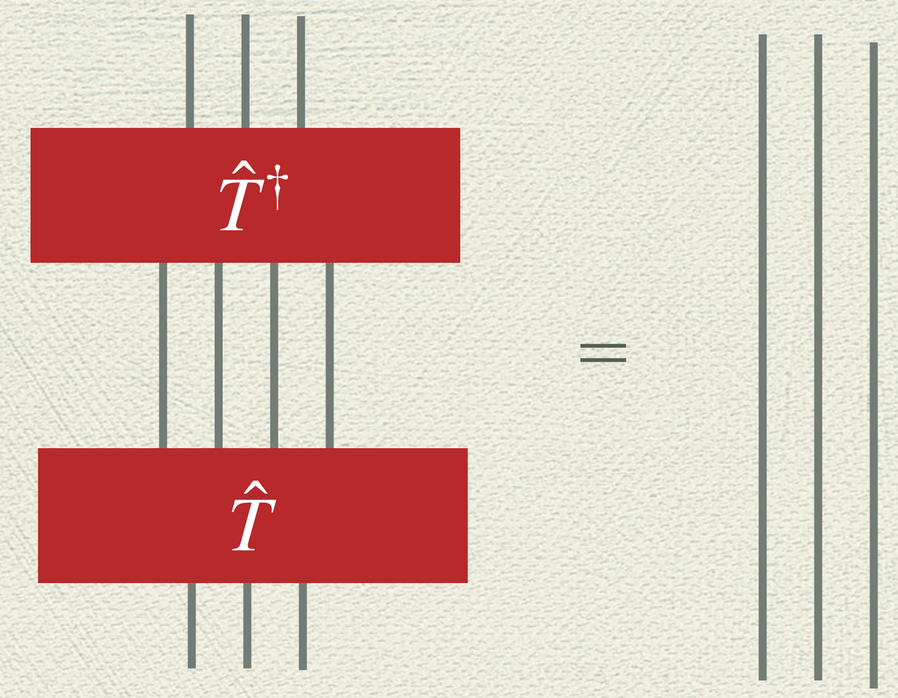




$\hat{T}$

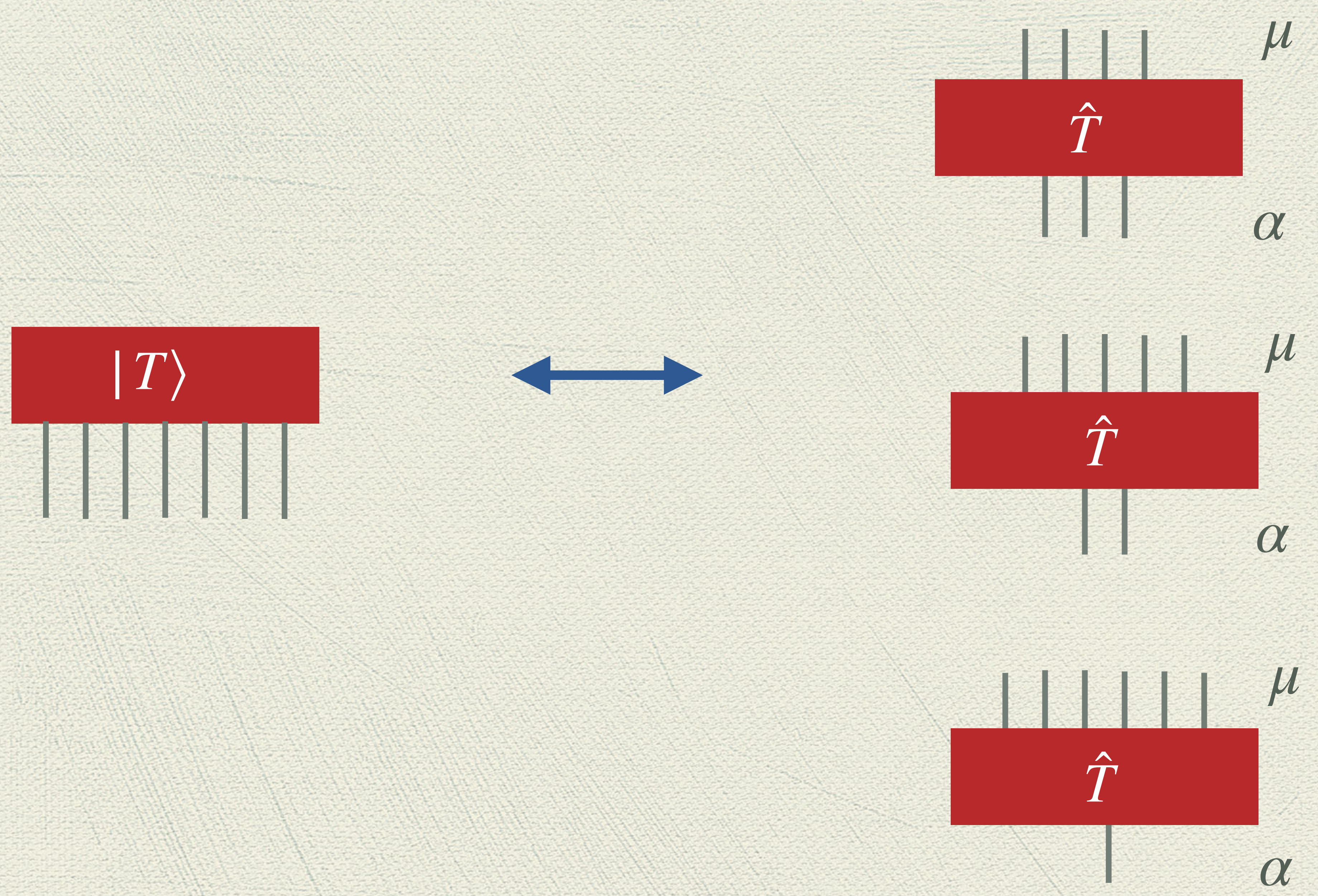


$\hat{T}^\dagger$



$\hat{T}^\dagger$
$\hat{T}$
=

$$|T\rangle = \sum_{\mu,\alpha} T_{\mu,\alpha} |\mu, \alpha\rangle \qquad \hat{T} = \sum_{\mu,\alpha} T_{\mu,\alpha} |\mu\rangle\langle\alpha|$$



We now show that this is all related to

Quantum State Sharing

As the simplest example, consider the first QSS scheme that we learnt:

$$|0\rangle \longrightarrow |\bar{0}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|1\rangle \longrightarrow |\bar{1}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

We can make an isometry out of this as follows:

$$\hat{T}|0\rangle = |\bar{0}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\hat{T}|1\rangle = |\bar{1}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

In matrix form:

$$\hat{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{T}^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

And we see that:

$$\hat{T}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad \hat{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{T}^\dagger \hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{T}\hat{T}^\dagger \neq I$$

Now we consider the general relation:

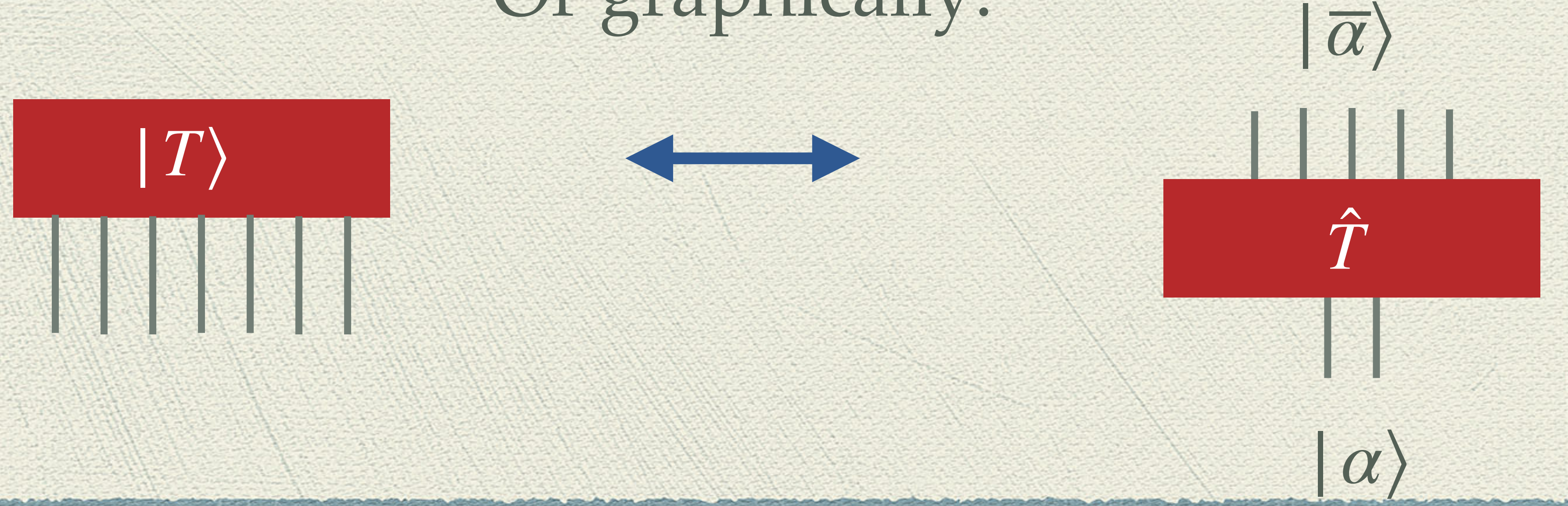
Let T be a perfect tensor or $|T\rangle$ an AME.

$$|T\rangle = \sum_{\mu,\alpha} T_{\mu,\alpha} |\mu, \alpha\rangle$$

Form the isometry: $$\hat{T} = \sum_{\mu,\alpha} T_{\mu,\alpha} |\mu\rangle\langle\alpha|$$

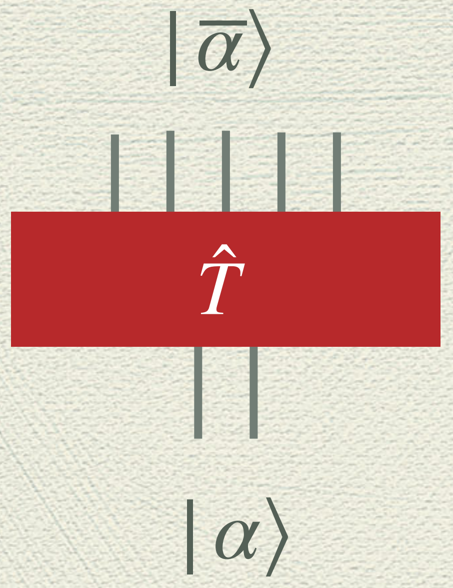
Which maps: $$\hat{T}|\alpha\rangle = |\overline{\alpha}\rangle$$

Or graphically:



We want to show that the map

$$\hat{T}|\alpha\rangle = |\overline{\alpha}\rangle$$

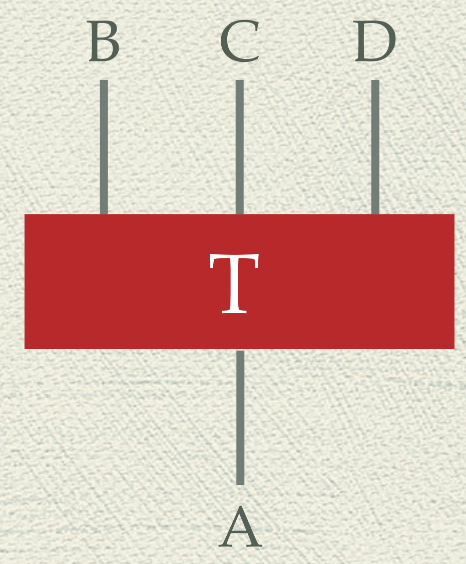


Is a quantum state sharing.

We can show this at least for a special class:

When one state is shared between 2n+1 parties,
so that none of the n-parties subsets can recover the state.

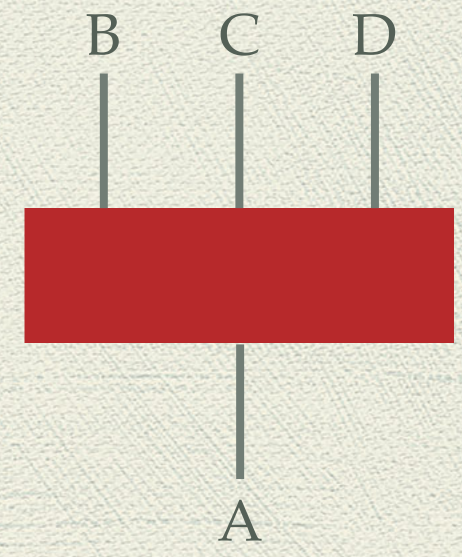
Consider the following example:



Since T is a perfect tensor,

T has the property that: $T_{ijkl}T_{i'j'kl} \propto \delta_{ii'}\delta_{jj'}$

A shares the basis states to BCD as follows:



$$|i\rangle_A \longrightarrow |\bar{i}\rangle_{BCD} = \sum_{j,k,l} T_{ijkl} |j,k,l\rangle_{BCD}$$

Therefore any state is shared as follows:

$$|\psi\rangle_A \longrightarrow |\overline{\psi}\rangle_{BCD}$$

Where

$$\overline{\psi}_{jkl} = \psi_i T_{ijkl}$$

The density matrix of B (or C or D)
is now proved to be maximally mixed,
hence B has no information about the shared state.

$$\overline{\psi}_{jkl} = \psi_i T_{ijkl}$$

$$(\rho_B)_{j,j'} = \overline{\psi}_{jkl}\overline{\psi}^*_{j'kl} = \psi_i T_{ijkl} \quad \psi^*_{i'} T_{i'j'kl}$$

$$= \psi_i \psi^*_{i'} \quad T_{ijkl} T_{i'j'kl}$$

$$\propto \psi_i \psi^*_{i'} \quad \delta_{ii'} \delta_{jj'} \ \propto \delta_{jj'}$$

The same is true for a perfect tensor of rand 2n.

We can use such a tensor to share a state between 2n-1 parties.

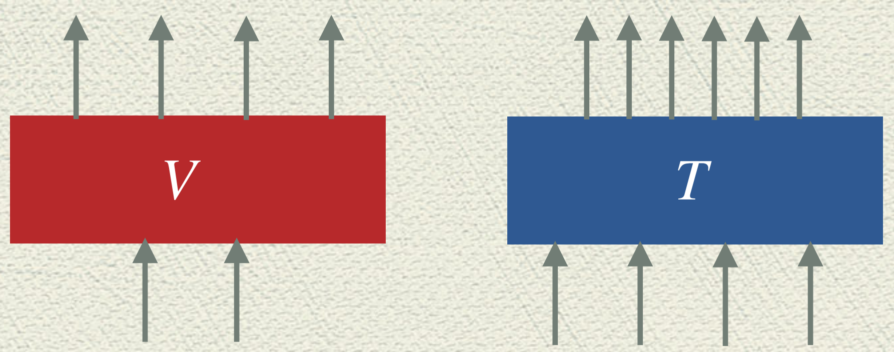
The density matrix of any group of n-1 members turns out to be proportional to I.

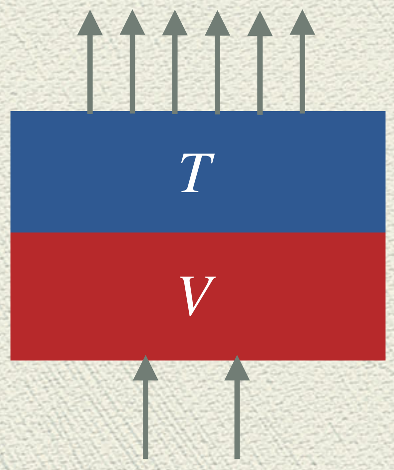
But I am sure that more general sharing schemes are possible, but I don't know how.

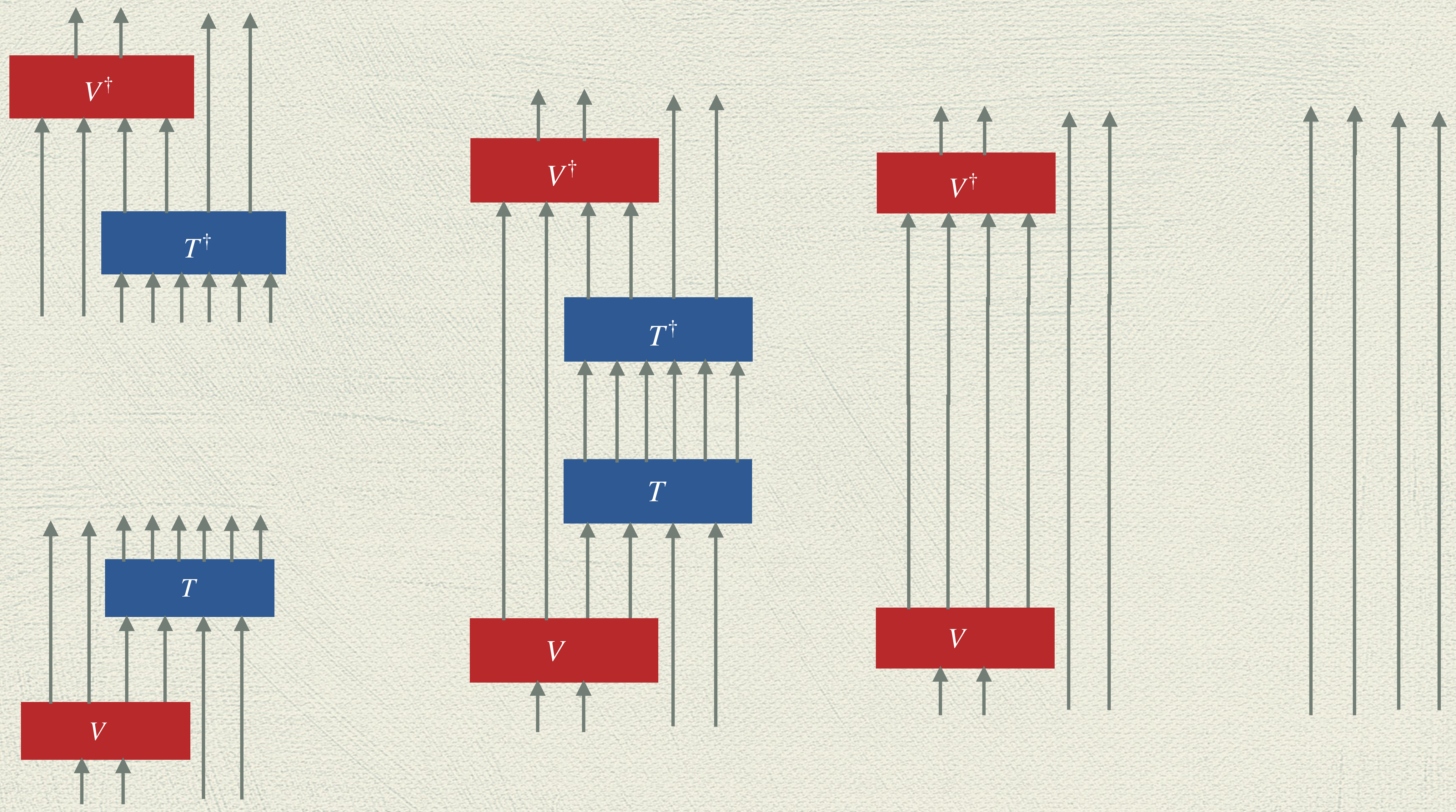
Combining Isometries

Let T and V be isometries, Then TV is also an isometry.

$$(TV)^{\dagger}TV = V^{\dagger}T^{\dagger}TV = I$$

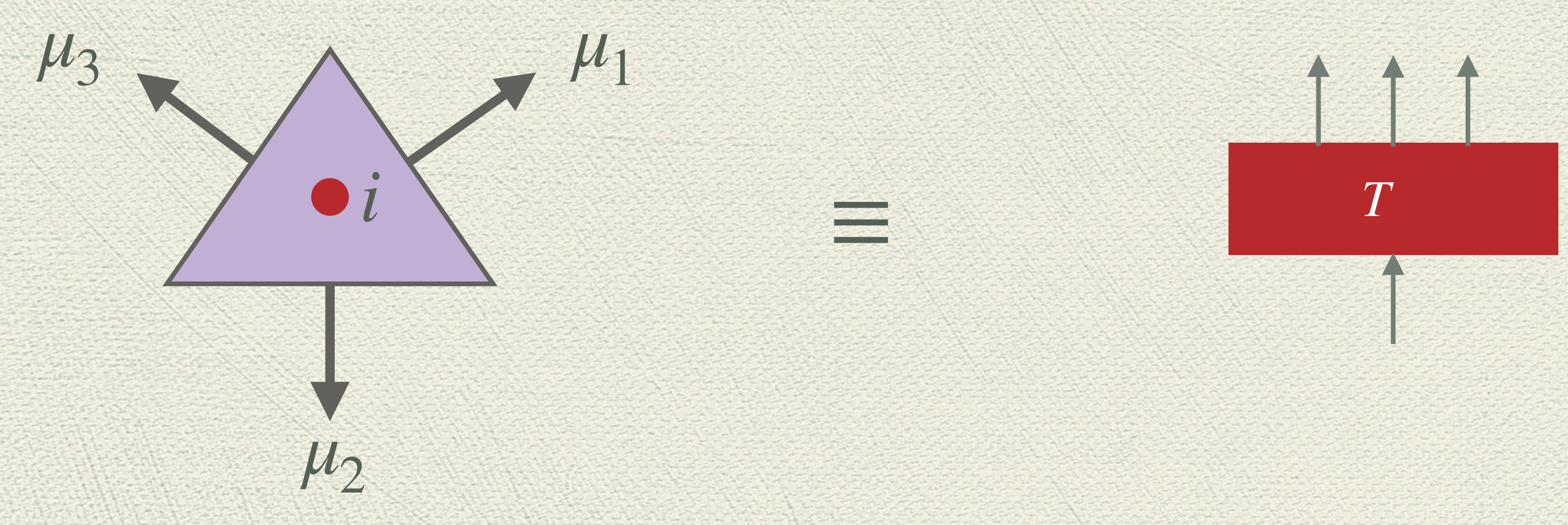




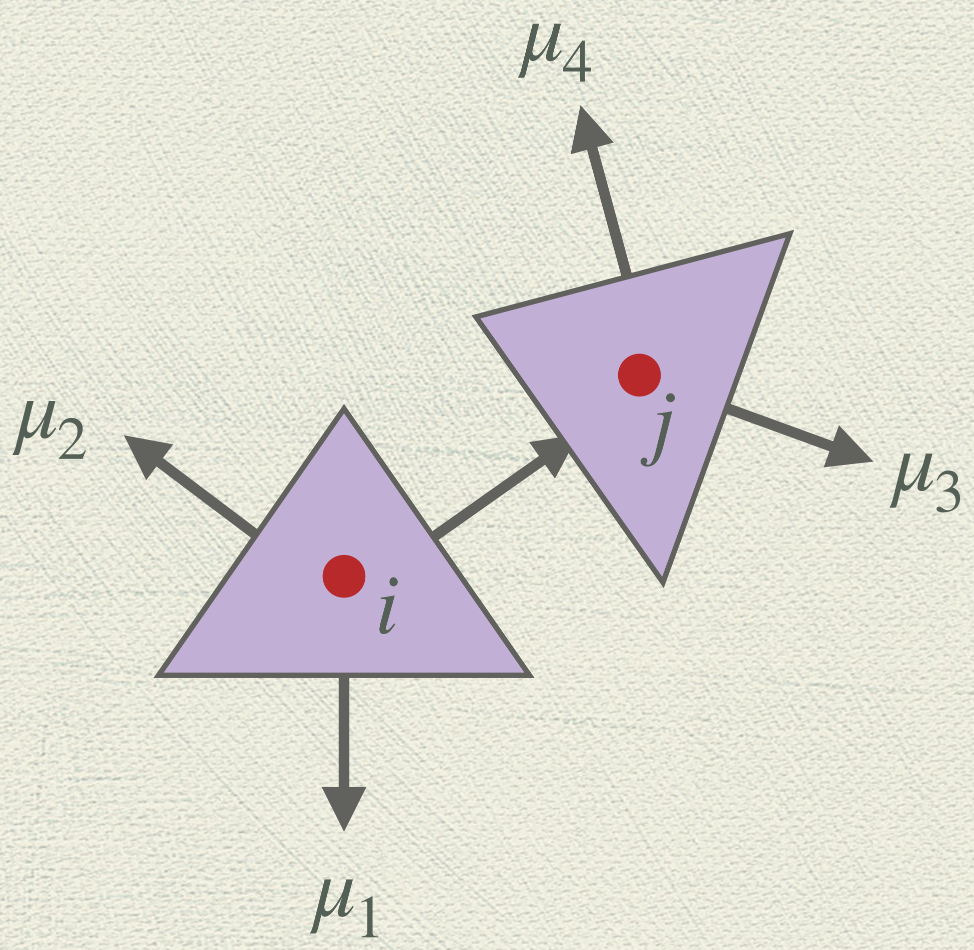

V†
T†
T
V
V†
T†
T
V
V†
V

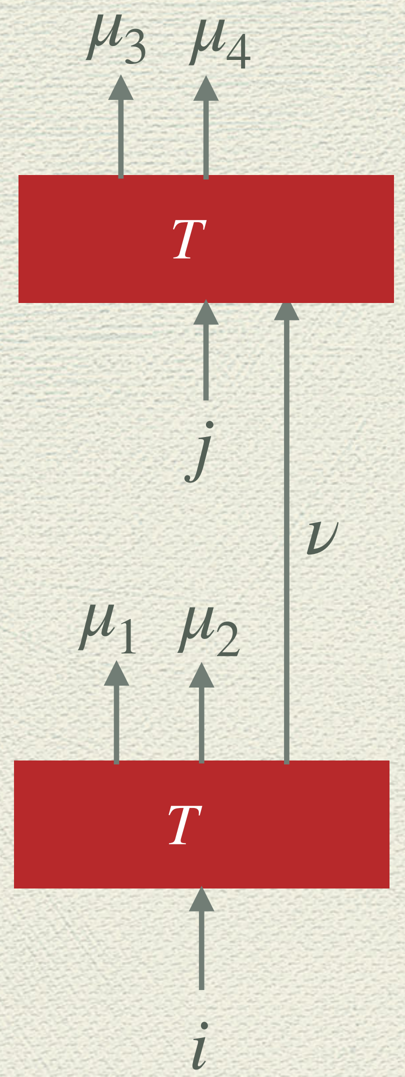
Tilings

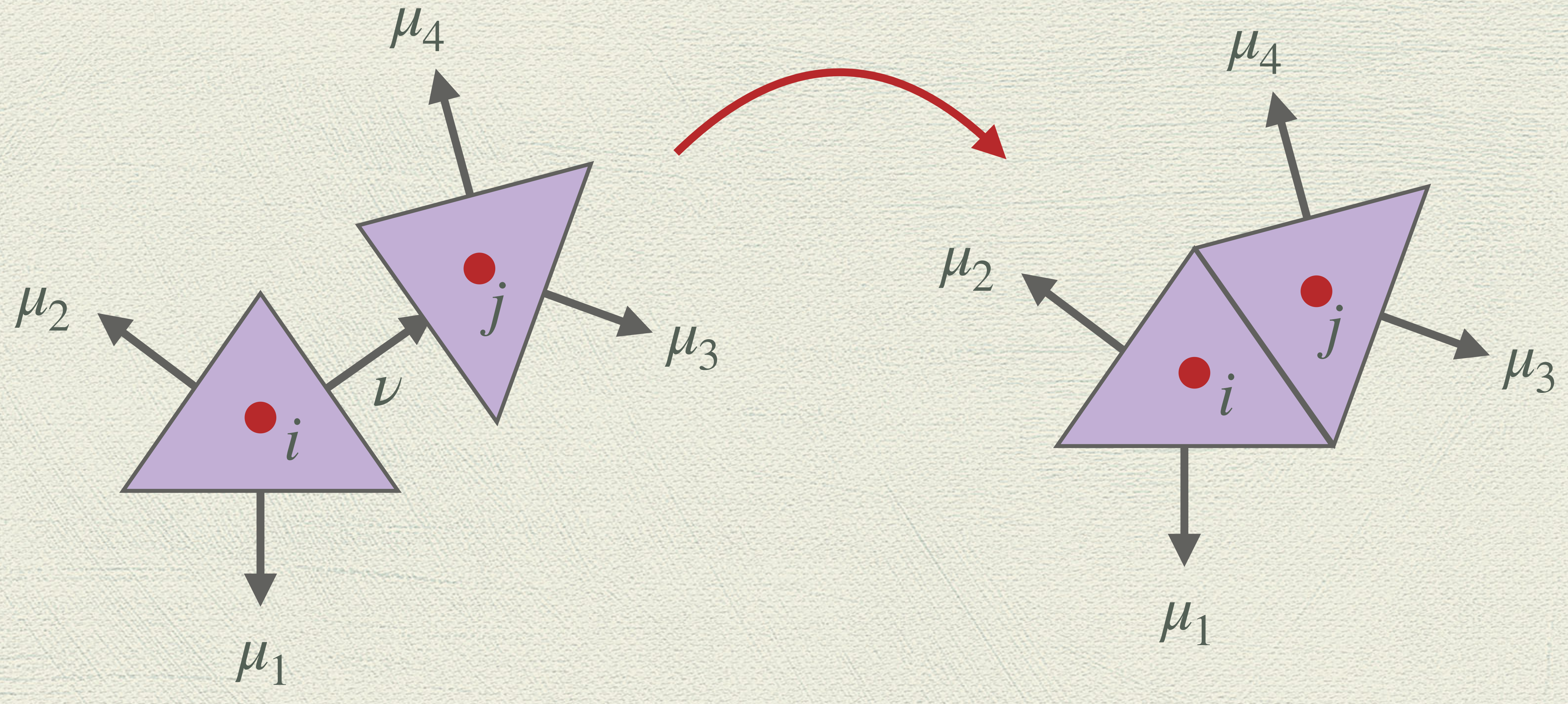
We can glue isometries to make a tiling of the plane



$$T_{i,\mu_1,\mu_2,\mu_3}$$


μ4
μ2
j
μ3
i
μ1

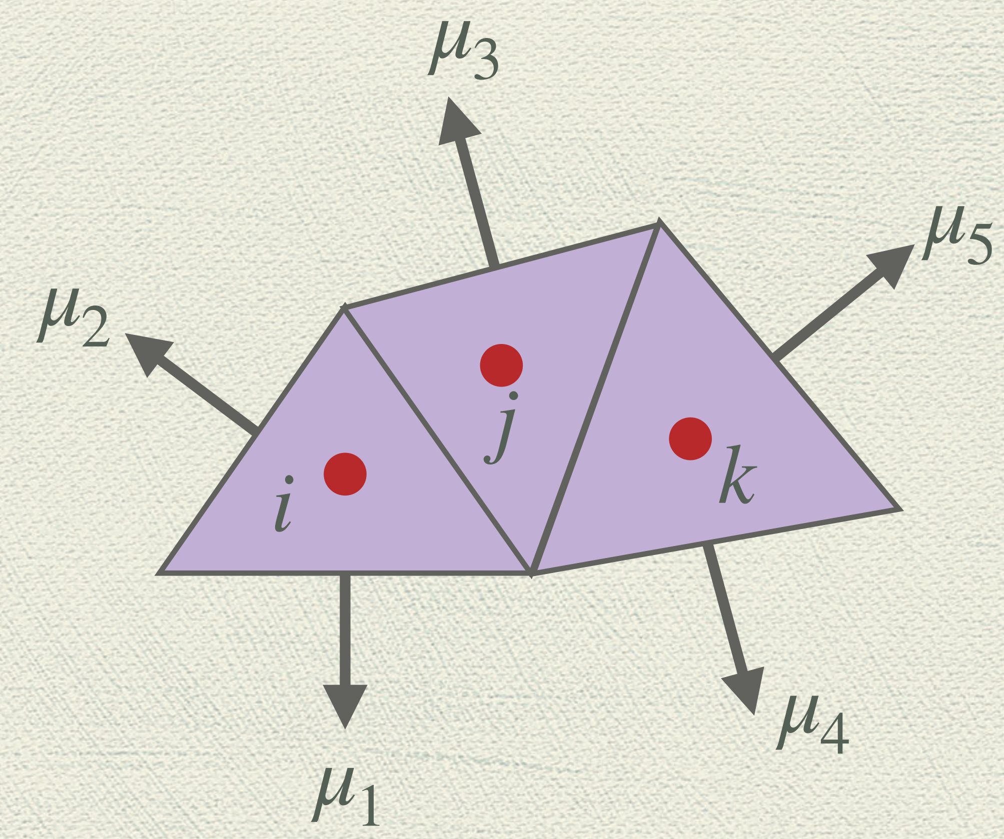

μ3 μ4
T
j
ν
μ1 μ2
T
i



From parts of the boundary states, we can recover the bulk state.

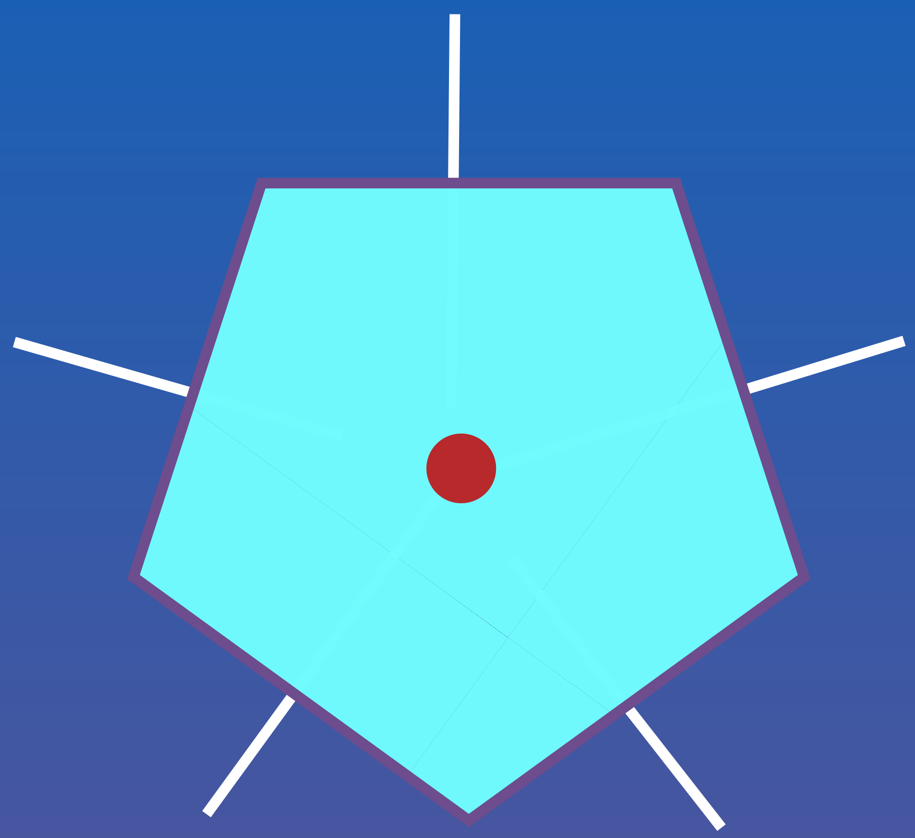
$$(\mu_1, \mu_2)\ \mu_3 \longrightarrow (i)\ \mu_3 \longrightarrow (\nu)\ \mu_3 \longrightarrow j$$

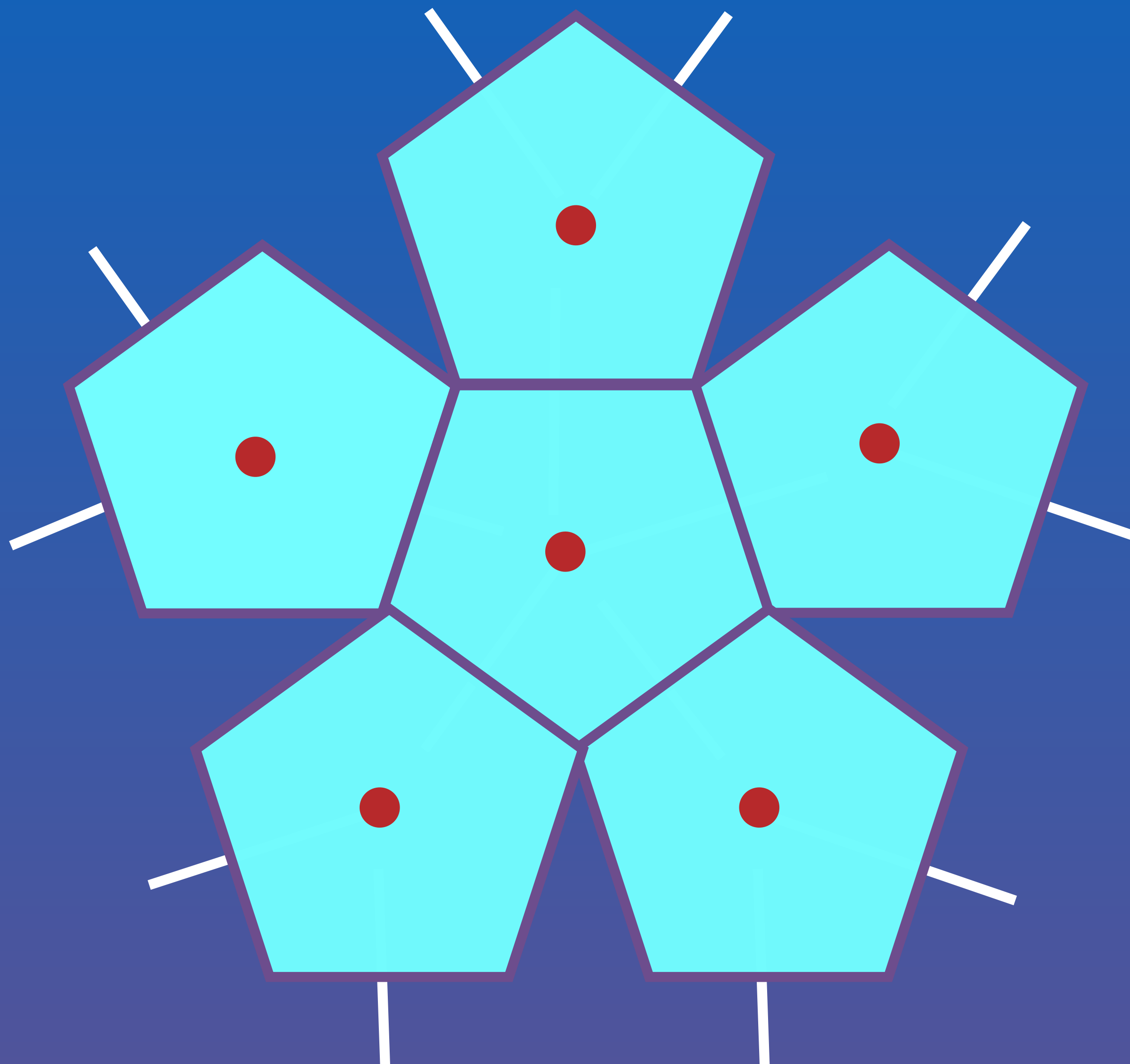
So i and j are retrieved.

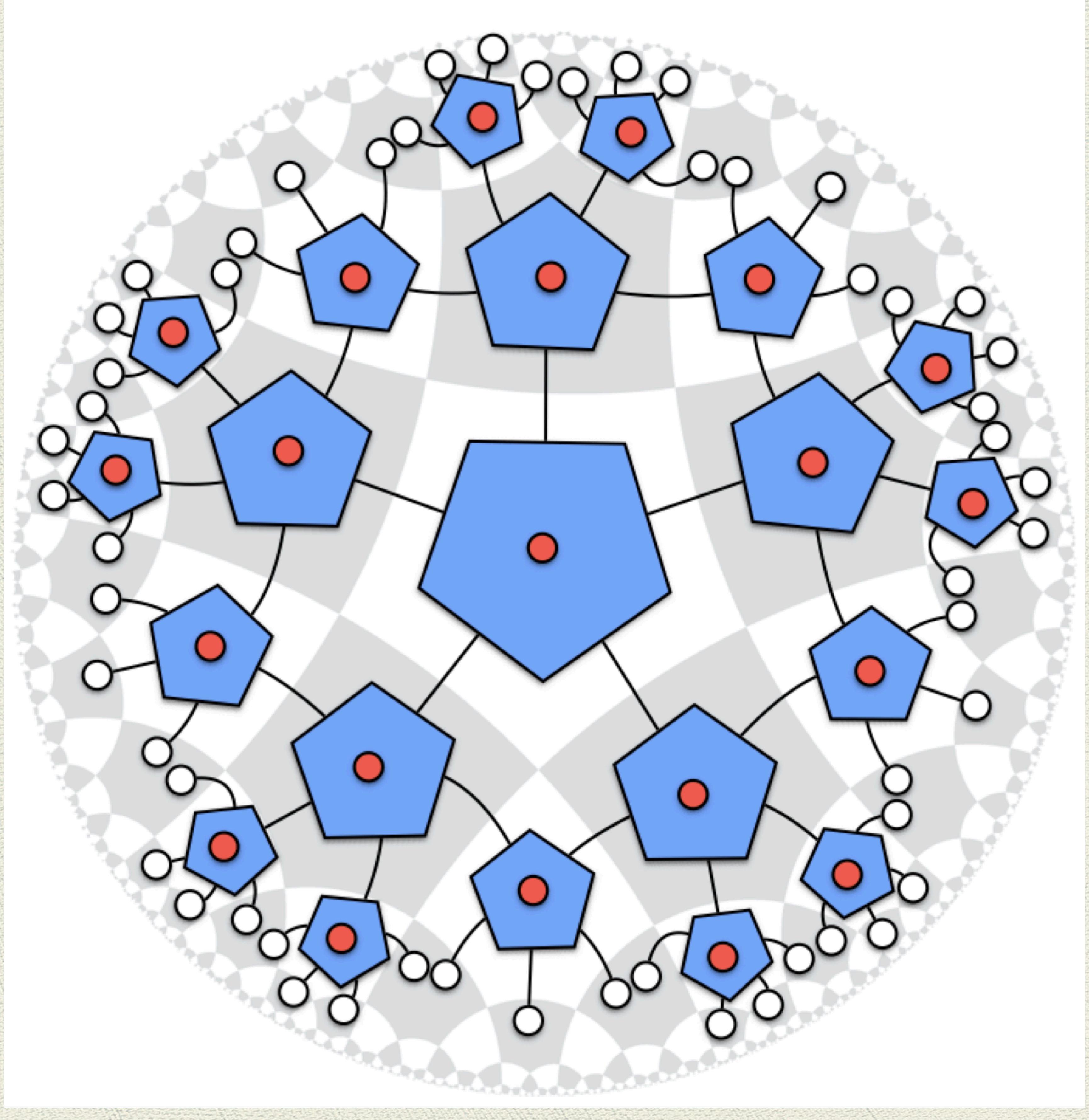


$$\mu_1, \mu_2, \mu_3, \mu_4 \longrightarrow i, j, k, \mu_5$$

More general tilings are possible.

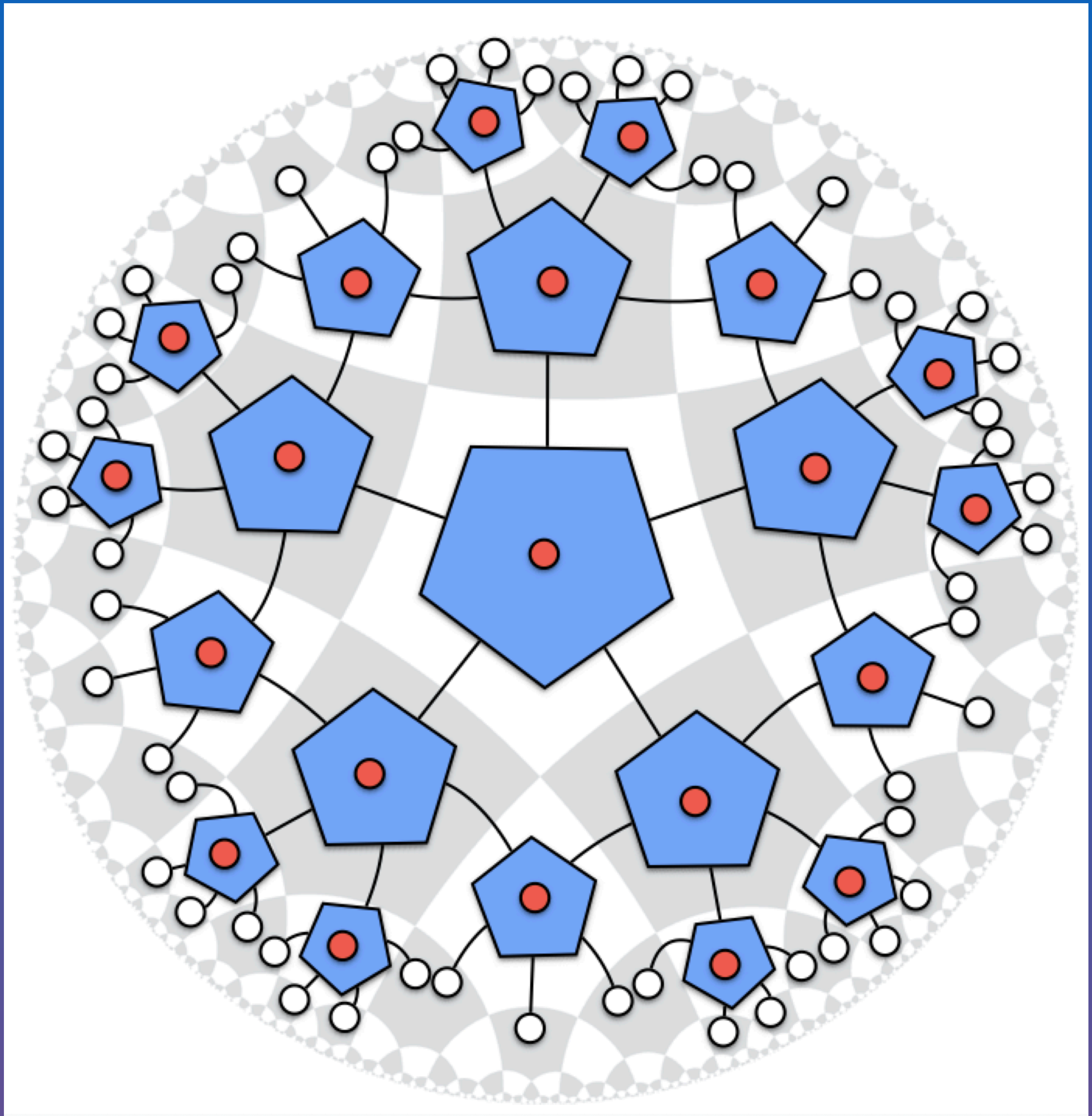






We can construct quantum states in which all the bulk information is encoded on the surface

There are similarities with the AdS/CFT Correspondence.



There are many un-answered questions:

How the metric becomes that of the Poincare plane?

Why this tiling becomes a hyperbolic tessellation?

What kinds of isometries lead to hyperbolic tessellation?

How this leads to bulk-boundary relation as we see in AdS/CFT?

How entanglement entropy enters this picture?

How black hole enters this picture?

End of part III